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Efficiency and the Value of Money

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In a monetary model, it is shown that if there is a unique Pareto inefficient barter equilibrium, then a monetary equilibrium exists when traders are sufficiently patient.

1. INTRODUCTION

This paper examines a model in which agents live forever, and money is the only asset. It shows that if consumers are sufficiently patient and inflation sufficiently low then there will be an equilibrium in which money has value. It also shows that when this sufficient condition is satisfied the non-monetary equilibrium is unstable in the sense that any small amount of government backing will cause the price of money to jump up from zero by a discrete amount.

This is a variant of a theorem due to Bewley (1980, 1983). Bewley showed that if endowments are "sufficiently" diverse, and the subjective discount rate is low enough, but greater than the rate of return on money (zero in this paper), then there is an equilibrium in which money has value. If we add two special but reasonable assumptions: positive transition probabilities between states, and a unique equilibrium in which money has no value, then this result can be greatly strengthened. The (rather complicated) assumption of "sufficient" diversity of endowments can be replaced by the simple requirement that the non-monetary equilibrium is not Pareto efficient. This can be either because tastes or endowments are (a little) diverse.

The relationship between the efficiency of barter equilibrium and existence of a monetary equilibrium has been extensively investigated in the overlapping-generations model. Indeed, Woodford (1986) has shown that in certain simple cases the conditions for the existence of a monetary equilibrium are the same in the overlapping-generations model as in the infinitely-lived agent, debt-constrained model. In the general overlapping-generations model, the best results are for the steady-state case, where Grandmont and Laroque (1973) show that inefficiency of non-monetary equilibrium implies existence of an efficient monetary steady state. Grandmont and Younes (1972) prove a similar result in a model with patient infinitely-lived agents and a cash-in-advance constraint, although the monetary steady-state is not efficient in this model. The best results in the general (non-steady state) overlapping-generations case are due to Burke (1985) who shows that inefficiency of barter implies the existence of an almost-efficient monetary equilibrium or of an efficient almost-monetary equilibrium. Millan (1985) has proven stronger results, but in a more limited model. Cass, Okuno, and Zilcha (1979) have some counter-examples showing that there is a limit to the extent to which these results can be strengthened.

One important difference between the overlapping-generations model and the infinite-lived debt-constrained model deserves note: in the overlapping-generations case the monetary equilibria are typically efficient, while Bewley (1980) shows that in the infinite-lived debt-constrained model without inflation monetary equilibria are typically not efficient. The cash-in-advance model is similar to the debt-constrained model in this regard. Grandmont and Younes (1973) show that the monetary steady state in this model is constrained inefficient. In the debt-constrained model with inflation, however, there may be efficient monetary equilibria: this point is made in Levine (1987).

The second section of the paper sets up the model; the third gives a statement and discussion of the main results. The fourth and fifth sections are devoted to proving the substantive theorems. Section six discusses the instability of non-monetary equilibria.

2. THE MODEL

All information is publicly held and common knowledge. Information at time t about current and future conditions is indexed by the state $\eta_t \in I$. There are only finitely many states in I. Information states $\{\eta_t\}$ naturally form a Markov chain; the initial probability distribution is fixed by the transition probabilities induced by the historically given initial state η_1. The transition probability from η to η' is denoted $\pi(\eta'|\eta)$. We always assume

Assumption 1. $\pi(\eta'|\eta)>0$ for all $\eta', \eta \in I$.

A state history $s=(\eta_1, \eta_2, \ldots, \eta_t)$ lists the states that have occurred up to and including time $t(s)\equiv t$. Corresponding to s is the state history $s-1$ which truncates s one period early at $t(s)-1$. If $t(s)=1$, so $s=\{\eta_1\}$, it is notationally convenient to define $s-1$ to be the null state history denoted 0. A state history s occurs with probability π_s induced by η_1 and the Markov transition probabilities $\pi(\eta'|\eta)$. The state history s also determines its final state $\eta_s \equiv \eta_{t(s)}$.

Agents are infinitely-lived, and goods are completely perishable; there are finitely many agents and goods. A typical agent is denoted a, and the finite set of all agents is A; a typical good is denoted ω, and the finite set of all goods is C. Let $x^a_\omega(s)$ be agent a's planned holding of good ω when the state history is s, and let $x^a(s)$ denote the corresponding column vector. Agent a's preferences are given by the utility function

$$\sum_s \delta^{t(s)} \pi_s u^a(x^a(s), \eta_s).$$

Note that δ is a common subjective discount factor used by all types. The one-period von Neuman–Morgenstern utility function $u^a(x^a(s), \eta_s)$ satisfies.

Assumption 2. For fixed $\eta \in I$, $u^a(x^a, \eta)$ is continuously differentiable, weakly concave, strictly monotone and bounded; the partial derivatives with respect to x^a_ω, denoted $D_\omega u^a(x^a, \eta)$ are bounded (even on the boundary of the consumption set).

The differentiability assumption can be weakened with only notational complication from the use of subgradients; dropping the assumption that the utility function has finite slope at the boundary leads to substantial complications. The assumption that u^a is bounded ensures that preferences are "continuous at infinity" in the sense of Fudenberg and Levine (1983), and makes it possible to draw inferences about infinite-horizon equilibrium from the finite-horizon one.

Agent a's endowment of good ω for each history s depends only on the state, and is denoted $z^a_\omega(\eta_s)$. We assume that endowments of goods are strictly interior.

Assumption 3. $z^a_\omega(\eta) > 0$ for all $\omega \in C$, $a \in A$, $\eta \in I$.

Consumption in each period must be nonnegative, so $x^a(s) \geqq 0$. The consumption good is completely perishable, so an allocation of goods is socially feasible if

$$\textstyle\sum_{a\in A} x^a(s) \leqq \sum_{a\in A} z^a(\eta_s) \equiv z(\eta_s). \tag{E.1}$$

If there is no intertemporal trade, the perishability of goods, and additive separability of preferences implies that the economy breaks up into countably many different finite pure exchange economies indexed by η_s. Specifically we define the *isolated economy* at η to be the finite pure exchange economy with commodities $\omega \in C$, endowments $z^a(\eta)$ and utility functions $u^a(x^a, \eta)$. Because it leads to substantially simpler and stronger results, we assume

Assumption 4. For each η the isolated economy has a unique competitive equilibrium.

An important case in which this assumption holds is when there is only one consumption claim, in which case the unique equilibrium is autarkic. Note that the "one" good may actually represent different commodities in different states.

In this economy the only asset is money: durable but intrinsically worthless certificates issued by the government. Let $M^a(s)$ be agent a's planned holding of money contingent on s. Since money cannot be forged, individual holdings must be nonnegative: $M^a(s) \geqq 0$. The initial aggregate money stock is $M(0)$, and it grows at the fixed rate $f \geqq 0$. An allocation of money is socially feasible if for every state history

$$\textstyle\sum_{a\in A} M^a(s) \leqq M(s) \equiv M(0)(1+f)^{t(s)}. \tag{E.2}$$

The impact of money growth on budget constraints depends on how the money is injected into the economy. If each agent receives new money in proportion to existing holdings, then the only effect is to change the units in which money is measured. We assume that money is injected into the economy in a lump sum manner so that each agent a receives $f^a M(s-1)$ dollars, where $\sum_{a\in A} f^a = f$, and $f^a \geqq 0$. Letting the prices of goods and money be denoted $p_\omega(s)$ and $p_M(s)$ respectively, it follows that if initial money holdings of agent a are $M^a(0)$ the corresponding budget constraints are

$$\begin{gathered} p_M(s)[M^a(s) - M^a(s-1) + f^a M(s-1)] + \textstyle\sum_{\omega\in C} p_\omega(s)[x^a_\omega(s) - z^a_\omega(\eta_s)] \leqq 0 \\ x^a(s) \geqq 0, \qquad M^a(s) \geqq 0. \end{gathered} \tag{E.3}$$

An *equilibrium* is a price vector, an allocation which is optimal at those prices for every type of agent subject to the constraint (E.3), and which satisfies the social feasibility conditions (E.1) and (E.2).

3. BARTER AND MONETARY EQUILIBRIA

There is always an equilibrium of this economy in which $p_M(s) \equiv 0$ and money has no value. We refer to this as a *barter equilibrium*, to emphasize the fact that in such an equilibrium goods may be traded for other types of goods but money may not be used to carry out intertemporal transactions. If $p_M(s) > 0$ for some state s, so that money at least sometimes has value, we say that the equilibrium is a *monetary equilibrium*. The main result of this paper is to show that if the barter equilibrium is inefficient, agents are sufficiently patient, and the growth rate of money is low enough, a monetary equilibrium exists.

Define

$$\Delta \equiv \delta/(1+f) \tag{3.1}$$

to be the inflation adjusted discount factor. Our main theorem is

Proposition 3.1. *Under Assumptions* 1-4 *if the (unique) barter equilibrium is not Pareto efficient, then there exists* $0<\underline{\Delta}<1$ *and a* $\underline{p}_M>0$ *such that for* $1>\Delta\geqq\underline{\Delta}$ *there is an equilibrium with the real value of the money stock relative to the real value of goods bounded below by* $\underline{p}_M$:

$$p_M(s)M(s)/\sum_{\omega\in C}p_\omega(s)z_\omega(\eta_s)>\underline{p}_M.$$

If the barter equilibrium is efficient there is no reason to trade between states, no need for money, and no reason to expect a monetary equilibrium to exist. On the other hand, it is easy to show that if Δ is sufficiently close to zero, no agent will be willing to forego consumption to hold money, and no monetary equilibrium will exist. Notice that for the hypothesis "Δ near one" to be satisfied, two conditions must be met: the subjective discount factor δ must be near one, and the inflation rate f must be near zero.

A problem with this result is that the monetary equilibrium is not the unique equilibrium: the barter equilibrium is also an equilibrium. It is natural to ask what forces tend towards the monetary equilibrium. Section 6 provides a partial answer to this question: under the assumptions of Proposition 3.1, if there is a small probability that the government redeems money for goods, then $p_M(s)M(s)/\sum_{\omega\in C}p_\omega(s)z_\omega(\eta_s)>\underline{p}_M$, where $\underline{p}_M>0$ is independent of the amount of redemption. In this sense, when barter is inefficient, agents patient and inflation low, barter is unstable: any small amount of redemption of money by the government causes the real value of the money stock to shoot up from zero to at least $\underline{p}_M$.

Related to this result is another, more paradoxical conclusion. Suppose there is a finite horizon, that the government redeems money at the end, and that the cost of holding money is low. Then the government can hold the current value of money fixed, while lowering the par value at the redemption date, provided that it moves the date of redemption further in the future. Consequently, two policies that might superficially appear to lower the current value of money—lowering the terminal value, and postponing the redemption date—in fact leave the current value of money unchanged.

Returning to Proposition 3.1, observe that the uniqueness of the barter equilibrium is an immediate consequence of Assumption 4: a barter equilibrium is equivalent to an equilibrium of the isolated economy after each finite history, and these are assumed to be unique. Specifically, if $x^a(\eta)$, $\{p_\omega(\eta)\mid\omega\in C\}$ are the unique allocation and price vector of the isolated economy at η (normalized so that $\sum_{\omega\in C}p_\omega(\eta)=1$), then the unique barter equilibrium satisfies $x^a(s)=x^a(\eta_s)$ and $p_\omega(s)=p_\omega(\eta_s)$. The next two sections are devoted to a proof of the remainder of Proposition 3.1. Before turning to these technical details, we first study the basic premise: that the barter equilibrium is not efficient.

To characterize efficiency, we associate with the unique equilibrium of the isolated economy at η, marginal utilities of expenditure $\mu^a(\eta)$; these are the Lagrange multipliers associated with individual optimization problems. Since by Assumption 3 endowments are strictly positive, in the barter equilibrium an agent of type a consumes a positive amount of some good, say ω. Moreover, by Assumption 2, the partial derivative of a's utility with respect to x^a_ω, is strictly positive. Consequently the price $p_\omega(\eta)$ is strictly positive and it follows that for good ω

$$\mu^a(\eta) = D_\omega u^a(x^a(\eta), \eta)/p_\omega(\eta), \tag{3.2}$$

which uniquely defines $\mu^a(\eta)$. Notice, incidentally, that since $D_\omega u^a(x^a(\eta), \eta) > 0$, $\mu^a(\eta) > 0$.

Let μ^a be the vector of a's marginal utilities of expenditure across states. Inefficiency of the barter equilibrium arises because the μ^a's are not proportional:

Lemma 3.2. *Under Assumptions* 2-4 *if the barter equilibrium is inefficient then there are two agents a and b such that* μ^a *and* μ^b *are not proportional.*

Proof. We show that proportionality of μ^a and μ^b for all agents a and b implies efficiency. Maximization of $\sum_{a\in A} \gamma^a U^a(x)$ subject to $x^a(s) \geqq 0$ and $\sum_{a\in A} x^a(s) \leqq \sum_{a\in A} z^a(\eta_s)$ for some non-negative weights γ^a is well-known to imply efficiency. In state s, the sufficient first-order condition is that there exist multipliers $q_\omega(s)$ such that

$$\gamma^a D_\omega u^a(x^a(s), \eta_s) \leqq q_\omega(s)$$

with complementary slackness. Observe that in a barter equilibrium

$$D_\omega u^a(x^a(s), \eta_s) \leqq \mu^a(\eta_s) p_\omega(\eta_s)$$

also with complementary slackness. Fix η, and set $\gamma^a = 1/\mu^a(\eta)$. If we define $q^a_\omega(s) = p_\omega(\eta_s)\mu^a(\eta_s)/\mu^a(\eta)$, proportionality of μ^a and μ^b implies $q^a_\omega(s) = q^b_\omega(s) = q_\omega(s)$, so that the sufficient condition for efficiency follows. ||

There are two reasons why barter equilibria may be inefficient: there can be diversity in tastes between states, or diversity in endowments between states. For simplicity, imagine there are two agents, two states and each agent is endowed with a single unit of the single consumption good. Suppose η has two components, η^1 for agent 1 and η^2 for agent 2, and for $x^a \leqq 2$, $u^a(x^a, \eta) = \eta^a x^a_\omega$ where ω is the single consumption good. In state 1, $\eta^1 = \bar{\eta}$, $\eta^2 = \underline{\eta}$, while in state 2, $\eta^1 = \underline{\eta}$, $\eta^2 = \bar{\eta}$. In this example, $\mu^a(\eta) = \eta^a$, and as long as $\bar{\eta} > \underline{\eta}$, Lemma 3.2 implies that the non-monetary equilibrium is inefficient. This is an example of diversity in tastes: agent 1 prefers to consume when he has the high marginal utility of consumption at $(\bar{\eta}, \underline{\eta})$ and agent 2 prefers $(\underline{\eta}, \bar{\eta})$.

An alternative model is to assume that tastes are identical and state-independent, while states are indexed by endowments of the consumption good. Specifically, suppose that η has two components, z^1, agent one's endowment, and z^2, agent two's endowment. In state 1, $z^1 = \bar{z}$, $z^2 = \underline{z}$, while in state 2, $z^1 = \underline{z}$, $z^2 = \bar{z}$. Consequently

$$\mu^a(\eta) = D_\omega u(z^a(\eta))$$

where u is the common state-independent utility function. Provided u is strictly concave and $\bar{z} > \underline{z}$, Lemma 3.2 again implies that the non-monetary equilibrium is inefficient.

The two-agent, two-state, one-good model with diverse endowments is closely related to the paper of Scheinkman and Weiss (1985). Their model differs from that here in several respects. They assume that utility is logarithmic, thus violating Assumption 2. They also use a continuous-time model in which states change according to a Poisson process. Their model may be viewed as a limiting case of ours in which the probability of changing states approaches zero, while the discount factor δ approaches one and the lower bound on utility approaches $-\infty$.

A more general condition guaranteeing diversity of endowments is given by Bewley (1980, 1983). He assumes that every individual has a state in which he is "poor" in the sense of having a very small endowment of all consumption goods (Bewley (1983), Assumption 3.8). He then gives a sufficient condition on preferences to guarantee that any "rich" individual, with income at least equal to that of the average agent, has marginal utility of expenditure strictly lower than that of any poor agent (Bewley (1983), Assumption 3.9 and (1980), Lemma 6). Fix a state η, and consider an agent, a, who has the least marginal utility of expenditure in that state: this person must be rich. Moreover, there must be another state η' in which he is poor. However, at η', some other agent, b, is rich and consequently has a strictly lower marginal utility of expenditure. Evidently μ^a and μ^b are not proportional, and Lemma 3.2 implies the nonmonetary equilibrium is inefficient.

Bewley's result on the existence of a monetary equilibrium is more general than Proposition 3.1 in several respects. He does not require a positive probability of all states (Assumption 1) nor uniqueness (Assumption 4). Moreover, he allows the possibility that there is a nominal rate of return on money financed by lump sum taxes. This is equivalent to a negative value of the inflation rate f. (Bewley does not explore positive rates of inflation.) Bewley's (1983) Theorem 3.10 shows that if the nominal return on money is not too large, and his endowment/taste assumptions hold, then a monetary equilibrium exists for δ sufficiently close to one.

In our context, Bewley's conditions are much too strong. It would be easy to show that for generic tastes and endowments, the barter equilibrium is inefficient: Lemma 3.2 says only that there must be *some* diversity in tastes and/or endowments across states.

4. FURTHER IMPLICATIONS OF INEFFICIENCY

Our first step in proving Proposition 3.1 is to develop as a corollary to Lemma 3.2 the fact that inefficiency implies the existence of stationary prices such that some agent would value money in every state. We also show that a similar property is true if the price of money is close to, rather than exactly equal to, zero. The actual proof of Proposition 3.1 is in the next section.

First we show

Lemma 4.1. *Under Assumptions* 1-4 *if the barter equilibrium is inefficient, then there are numbers* $0<\gamma<1$, *and* $0<\underline{\Delta}<1$ *and for each* $\eta\in I$ *a scalar* $p_m(\eta)>0$ *and agent* $a(\eta)$ *such that*

$$p_m(\eta)\mu^{a(\eta)}(\eta)<\gamma\underline{\Delta}\sum_{\eta'\in I}\pi(\eta'|\eta)p_m(\eta')\mu^{a(\eta)}(\eta'). \tag{4.1}$$

Proof. By Lemma 3.2 we may assume that for agents a, b, and states η^a and η^b there are positive constants α^a, α^b such that

$$\alpha^a\mu^a(\eta^a)>\alpha^b\mu^b(\eta^a);\ \alpha^a\mu^a(\eta^b)<\alpha^b\mu^b(\eta^b).$$

Set $p_m(\eta)=[\min\{\alpha^a\mu^a(\eta),\alpha^b\mu^b(\eta)\}]^{-1}$, and let $a(\eta)$ be an agent who assumes the minimum. Then $p_m(\eta)\mu^{a(\eta)}(\eta)=1/\alpha^{a(\eta)}$, while $p_m(\eta')\mu^{a(\eta)}(\eta')\geqq 1/\alpha^{a(\eta)}$ with strict inequality for $\eta'=\eta^{a(\eta)}$. Since $\pi(\eta'|\eta)>0$ we conclude that

$$p_m(\eta)\mu^{a(\eta)}(\eta)<\sum_{\eta'\in I}\pi(\eta'|\eta)p_m(\eta')\mu^{a(\eta)}(\eta').$$

Since I is finite, this implies the right-hand side is larger than the left when multiplied by a constant $\gamma\underline{\Delta}$ sufficiently close to 1. ||

Next we wish to show that Lemma 4.1 continues to hold, provided money does not have too much value. Following Bewley (1980) we can define for $0 \leqq \alpha \leqq 1$ an *α-transfer payment equilibrium* of the isolated economy at η. This is an efficient allocation x which, when corresponding efficiency prices q are normalized to add up to $1-\alpha$, satisfies

$$\textstyle\sum_{\omega \in C} q_\omega [x^a_\omega - z^a_\omega(\eta)] \leqq \alpha.$$

If $\alpha = 0$ this is simply the unique Walrasian equilibrium of the isolated economy, discussed above. If we normalize prices so that $\sum_\omega p_\omega(s) + p_M(s)M(s) = 1$, since no agent can trade more than the entire money stock, any equilibrium allocation at s with $p_M(s)M(s) = \alpha$ is an α-transfer payment equilibrium of the isolated economy.

Associated with each α-transfer payment equilibrium x are the marginal utilities of expenditure for agents, $\mu^a(x)$; these are the inverses of the efficiency weights which give rise to the transfer payment equilibrium. Let $\bar{\mu}^a(\alpha, \eta)$ be the *supremum* of $\mu^a(x)$ for any α'-transfer payments equilibrium x with $\alpha' \leqq \alpha$, and let $\underline{\mu}^a(\alpha, \eta)$ be the *infimal* value. Note that we take the sup and inf over $\alpha' \leqq \alpha$, because prices are normalized differently for each α'. The key property of these bounds is that for small values of α, $\bar{\mu}$ and $\underline{\mu}$ are approximately equal and positive:

Lemma 4.2. *Under Assumptions* 1–4

$$\lim_{\alpha \to 0} \underline{\mu}^a(\alpha, \eta) = \lim_{\alpha \to 0} \bar{\mu}^a(\alpha, \eta) = \underline{\mu}^a(0, \eta) = \bar{\mu}^a(0, \eta) > 0.$$

Proof. If $\mu^a_n(\eta)$ are a convergent sequence of marginal utilities of expenditure for α_n-transfer payments equilibrium with $\alpha \to 0$ a straightforward continuity argument shows that their limit is a marginal utility of expenditure for a 0-transfer payments equilibria. It follows that

$$\underline{\mu}^a(0, \eta) \leqq \liminf_{\alpha \to 0} \underline{\mu}^a(\alpha, \eta) \leqq \limsup_{\alpha \to 0} \bar{\mu}^a(\alpha, \eta) \leqq \bar{\mu}^a(0, \eta).$$

On the other hand, we remarked above in (3.2) that when $\alpha = 0$, the marginal utilities of expenditure are uniquely determined and positive. Consequently $\underline{\mu}^a(0, \eta) = \bar{\mu}^a(0, \eta) > 0$, and this implies the desired conclusion. ||

An immediate consequence of Lemma 4.2 is that Lemma 4.1 continues to hold for $\alpha > 0$. This is similar in spirit to Bewley's (1980) Lemma 7.

Corollary 4.3. *Under Assumptions* 1–4, *suppose*

$$p_m(\eta)\bar{\mu}^a(\alpha, \eta) < \gamma\underline{\Delta} \textstyle\sum_{\eta' \in I} \pi(\eta' \mid \eta)\underline{\mu}^a(\alpha, \eta') \tag{4.2}$$

holds for $\alpha = 0$. *Then there is an* $\alpha > 0$ *for which it also holds.*

Proof. Follows directly from Lemma 4.2. ||

5. PROOF OF THE MAIN THEOREM

It is convenient and there is no loss of generality in measuring money in units that represent a share of the total money stock; we define

$$m^a(s) \equiv M^a(s)/M(s). \tag{5.1}$$

In these units the total money stock is one, and the social feasibility condition (E.2) becomes

$$\sum_{a\in A} m^a(s) \leqq 1, \tag{E.2'}$$

while the budget constraint is

$$\begin{gathered} p_m(s)\left[m^a(s)-\frac{m^a(s-1)+f^a}{1+f}\right]+\sum_{\omega\in C} p_\omega(s)[x^a_\omega(s)-z^a_\omega(\eta_s)]\leqq 0 \\ x^a(s)\geqq 0, \qquad m^a(s)\geqq 0. \end{gathered} \tag{E.3'}$$

The price normalization $\sum_{\omega\in C} p_\omega(s)+p_M(s)M(s)=1$, corresponds to

$$\sum_{\omega\in C} p_\omega(s)+p_m(s)=1$$

with $p_m(s)=p_M(s)M(s)$, that is, $p_m(s)$ now measures the real value of the money stock. Notice that if $p_m(s)$ is stationary, then $p_M(s)$ shrinks at the rate $1+f$; in other words, the inflation rate is $1+f$.

To prove Proposition 3.1, we introduce truncated equilibria. The truncated utility function for agent a is

$$U^a_T(x^a)=\sum_{t(s)\leqq T}\delta^{t(s)}\pi_s u^a(x^a(s),\eta_s)+\sum_{t(s)=T}\delta^T\pi_s\phi^a_s m^a(s). \tag{5.2}$$

The terminal multipliers ϕ^a_s are any numbers satisfying

$$0\leqq\phi^a_s\leqq\bar{\phi}, \tag{5.3}$$

that is, they are uniformly bounded. The rationale behind the terminal multipliers is that they make it possible to construct equilibria in which money has positive value from limits of truncated equilibria.

Given prices p the set of x^a, m^a that satisfy the constraints (E.3') and the initial condition is the budget set $B^a(p)$. There are also truncated budget sets $B^a_T(p)$ in which the constraints (E.3') are imposed only for states s with $t(s)\leqq T$. A *truncated equilibrium* is socially feasible and individually rational for the truncated utility functions with respect to the truncated budget constraints.

Proposition 5.1. *Under* (*A*.2) *to* (*A*.3) *every sequence of truncated economies has a convergent subsequence of truncated equilibria*; *every such subsequence converges to an equilibrium.*

Proof. See Levine (1985) and Bewley (1980). ||

Note that the hypothesis $f^a\geqq 0$ for all agents is important here. Bewley (1983) shows that if some of the f^a are too negative (so that there are lump sum taxes rather than subsidies), it may be that agents cannot pay their taxes, and the truncated economies may have no equilibrium.

We now review the necessary and sufficient first-order conditions for a truncated equilibrium. Associated with each budget constraint is the non-negative marginal utility of expenditure $\mu^a(s)$: the first-order condition for optimal consumption is

$$D_\omega u^a(x^a(s),\eta_s)\leqq\mu^a(s)p_\omega(s)\quad\text{for all }\omega\in C\text{ and }t(s)\leqq T \tag{5.4}$$

with equality unless $x^a_\omega(s)=0$. Recall that $\Delta\equiv\delta/(1+f)$ is the inflation-adjusted discount factor. The first-order condition for optimal money holding is

$$p_m(s)\mu^a(s)\geqq\Delta\sum_{\eta'\in I}\pi(\eta'\mid\eta_s)p_m(s,\eta')\mu^a(s,\eta'),\qquad t(s)<T \tag{5.5}$$

with equality unless money holding is zero. Finally, the terminal condition is

$$p_m(s)\mu^a(s) \geqq \phi^a_s, \qquad t(s) = T. \tag{5.6}$$

An alternative to truncated equilibria that exploits stationarity to compute infinite horizon equilibria is considered in Kehoe and Levine (1986).

In light of Lemma 4.1, in order to prove Proposition 3.1, it suffices to show

Proposition 5.2. *Under Assumptions* 2-4 *suppose that there is a* $0<\gamma<1$ *and* $0<\underline{\Delta}<1$, *and for each state* $\eta \in I$ *there are scalars* $p_m(\eta)>0$ *and an agent* $a(\eta)$ *such that*

$$p_m(\eta)\mu^{a(\eta)}(\eta) < \gamma\underline{\Delta} \textstyle\sum_{\eta' \in I} \pi(\eta' \mid \eta) p_m(\eta')\mu^{a(\eta)}(\eta'). \tag{5.7}$$

Then there is a $\underline{p}_m > 0$ *and for* $1 > \Delta \geqq \underline{\Delta}$ *an equilibrium in which* $p_m(s) > \underline{p}_m$.

Note that if $p_m(s) > \underline{p}_m$, then

$$p_m(s)/\textstyle\sum_{\omega \in C} p_\omega(s) z_\omega(\eta_s) > \underline{p}_m / \max_{\omega,\eta} z_\omega(\eta) \equiv \underline{p}_M,$$

giving the conclusion of Proposition 3.1.

Notice that (5.5) says that the current marginal value of expenditure must not be less than the future value, for otherwise agent a would choose to hold more money. Condition (5.7) says that there are stationary prices $p_m(\eta)$ depending only on the state, so that if money was introduced at those prices into a world without money, then agent $a(\eta)$ would strictly prefer to hold money. The conclusion of Proposition 5.2 is that in this case there is a monetary equilibrium, although it will generally involve prices different than $p_m(\eta)$ and different marginal utilities of income.

Proof of Proposition 5.2. We construct a sequence of truncated T-period equilibria with bounded terminal weights in which

$$p_m(s) \geqq \underline{p}_m > 0.$$

The theorem then follows from Proposition 5.1.

We begin by considering an arbitrary truncated T-period equilibrium: we let $\mu^a(s)$ be the corresponding Lagrange multipliers from (5.4) and (5.5), and let $p(s)$ denote the prices. We also let $D_\omega u^a(s) \equiv D_\omega u^a(x^a(s), \eta_s)$ be the equilibrium marginal utilities of consumption.

First observe that there is a $\underline{\mu} > 0$ such that $\mu^a(s) \geqq \underline{\mu}$. This is because (5.4) says $D_\omega u^a(s) \leqq \mu^a(s) p_\omega(s)$, while $p_\omega(s) \leqq 1$, implies $\mu^a(s) \geqq D_\omega u^a(s)$. Since u^a is strictly monotone and $0 \leqq x^a_\omega(s) \leqq z_\omega(\eta_s)$, $D_\omega u^a(s)$ is bounded uniformly away from zero, implying in a truncated equilibrium

$$\mu^a(s) \geqq \underline{\mu} > 0.$$

Next choose α so that (4.2) holds in each state η for $a(\eta)$. This is possible by Corollary 4.3. Also choose α so that $\underline{\mu}^a(\eta, \alpha) > 0$. This is possible by Lemma 4.2. Finally, choose $p_m(\eta)$ to satisfy (5.7) and such that

$$p_m(\eta) \leqq \underline{\mu}\alpha / \max_{a,\eta} \underline{\mu}^a(\eta, \alpha) \tag{5.8}$$

which is possible, since (5.7) is homogeneous.

Now let ψ be any non-negative scalar, and choose the terminal weights $\phi^a(s) = \psi p_m(\eta_s)\underline{\mu}^a(\eta_s, \alpha)$. Since these depend only on η_s, they clearly satisfy the transversality condition (5.3). Define $\gamma_{s-1} = \min(1, \psi\gamma^{t(s)-T})$. Our goal is to show that in any truncated T-period equilibrium with the terminal weights defined above

$$p_m(s)\mu^a(s) \geqq \gamma_{s-1} p_m(\eta_s)\underline{\mu}^a(\eta_s, \alpha) \quad \text{for all } a. \tag{5.9}$$

Our final step will use the boundedness of the $\mu^a(s)$'s to derive a bound on $p_m(s)$.

We show that (5.9) holds by backwards induction from the terminal period. In the final period (5.9) holds by the terminal first-order condition (5.6) and the definition of the weights. We carry out the inductive step by making use of (4.2). By inductive hypothesis, we see from (5.5) that

$$p_m(s)\mu^a(s) \geqq \gamma_s \Delta \textstyle\sum_{\eta' \in I} \pi(\eta' | \eta) p_m(\eta')\underline{\mu}^a(\eta', \alpha) \tag{5.10}$$

holds for all agents. Suppose for some agent a

$$p_m(s)\mu^a(s) < \gamma_{s-1} p_m(\eta_s)\underline{\mu}^a(\eta_s, \alpha). \tag{5.11}$$

Since $\mu^a(s) \geqq \underline{\mu}$ this implies by (5.8)

$$p_m(s) < \gamma_{s-1} p_m(\eta_s)\underline{\mu}^a(\eta_s, \alpha)/\underline{\mu} \leqq \alpha.$$

In other words, we chose $p_m(\eta)$ so small that, since $\mu^a(s) \geqq \underline{\mu}$, the bound can be violated only because $p_m(s) < \alpha$. Consequently, $x^a(s)$, $p_\omega(s)$ are an α-transfer payment equilibrium of the isolated economy, and since $\mu^a(s)$ are clearly the associated marginal utilities of expenditure, for all $a \in A$,

$$\bar{\mu}^a(\eta_s, \alpha) \geqq \mu^a(s) \geqq \underline{\mu}^a(\eta_s, \alpha).$$

We conclude from (5.11) that

$$p_m(s) < \gamma_{s-1} p_m(\eta_s).$$

It follows that,

$$p_m(s)\mu^{a(\eta)}(s) < \gamma_{s-1} p_m(\eta_s)\bar{\mu}^{a(\eta)}(\eta_s, \alpha).$$

Since (5.10) holds for $\alpha(\eta)$ we conclude

$$\begin{aligned} \gamma_{s-1} p_m(\eta_s)\bar{\mu}^{a(\eta)}(\eta_s, \alpha) &> p_m(s)\mu^{a(\eta)}(s) \\ &\geqq \gamma_s \Delta \textstyle\sum_{\eta' \in I} \pi(\eta' | \eta) p_m(\eta')\underline{\mu}^{a(\eta')}(\eta', \alpha). \end{aligned}$$

Since $\gamma_s/\gamma_{s-1} \geqq \gamma$, and $\Delta \geqq \underline{\Delta}$, this contradicts the fact that (4.2) was assumed to hold for $a(\eta)$.

Having demonstrated (5.9), we conclude by deriving a bound on prices. For each good ω, and history s, in a truncated equilibrium there is an agent $a(\omega, s)$ who consumes a positive amount of a good ω. For such an agent we have

$$D_\omega u^{a(\omega,s)}(s) = \mu^{a(\omega,s)}(s) p_\omega(s). \tag{5.12}$$

By Assumption 2, $D_\omega u^a$ is bounded above, say by $\bar{d}$. Adding (5.12) across goods, we have

$$|A|\bar{d} \geqq \textstyle\sum_{\omega \in C} D_\omega u^{a(\omega,s)}(s) \geqq (1 - p_m(s))|A| \min_{a \in A} \mu^a(s), \tag{5.13}$$

where $|A|$ is the number of agents, and we have used normalization $\sum_{\omega \in C} p_\omega(s) + p_m(s) = 1$. From (5.9), we have, for $a(\eta_s)$ chosen to minimize $\underline{\mu}^a(\eta_s, \alpha)$

$$\bar{d} \geqq \frac{(1 - p_m(s))}{p_m(s)} \gamma_{s-1} p_m(\eta_s)\underline{\mu}^{a(\eta_s)}(\eta_s, \alpha), \tag{5.14}$$

from which it follows that

$$p_m(s) \geqq \frac{\gamma_{s-1} p_m(\eta_s) \underline{\mu}^{a(\eta_s)}(\eta_s, \alpha)}{\bar{d} + \gamma_{s-1} p_m(\eta_s) \underline{\mu}^{a(\eta_s)}(\eta_s, \alpha)}. \tag{5.15}$$

If we choose $\psi \geqq 1$, since $\gamma < 1$, $\gamma_{s-1} = 1$ for all s. Moreover, we chose α so that $\underline{\mu}^a(\eta, \alpha) > 0$, while $p_m(\eta) > 0$ by assumption. Consequently the proposition follows from (5.15). ‖

6. INSTABILITY OF BARTER EQUILIBRIA

In the course of proving Proposition 5.2 we can, by making judicious use of the γ's, prove a stronger proposition about truncated equilibria. Suppose that the horizon is finite and the government redeems money at the end. Then there is a number ψ, determined by the redemption value of money, so that

$$\phi^a(s) \geqq \psi p_m(\eta_s) \underline{\mu}^a(\eta_s, \alpha).$$

The less the government redeems the money for, the smaller is ψ. Set $K = \log \psi / \log \gamma$, so that $\gamma^{-K}\psi = 1$. Then if $t(s) \leqq T - K$, $p_m(s) > \underline{p}_m$. In particular, the longer the horizon, the smaller the redemption value need be to insure a minimal value of money early on.

This line of argument also applies to the case where there is an infinite horizon, and a fixed probability of a fixed redemption each period. In this case

$$\textstyle\sum_{\eta' \in I} \pi(\eta' | \eta) p_m(\eta') \mu^{a(\eta)}(\eta')$$

should be interpreted as the sum over next period states η' in which redemption does not occur. Consequently $\sum_{\eta' \in I} \pi(\eta' | \eta) < 1$. However, if (5.7) held without redemption and the redemption probability is small enough, it will continue to hold with redemption. The argument in the proof of Proposition 5.2 is then unchanged, except that we may now assume that the terminal value of money is bounded away from zero by value of redemption. Consequently every equilibrium must satisfy $p_m(s) > \underline{p}_m$.

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