18 Steady States and Determinacy of Equilibria in Economies With Infinitely Lived Agents

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1 INTRODUCTION

Joan Robinson frequently argued that neoclassical general equilibrium theory could not determine the rate of interest in intertemporal models (see, for example, Robinson, 1973). There were two aspects to this critique: First, neoclassical marginal productivity theory depended on the notion of an aggregate capital stock. Because of aggregation problems, notably reswitching, this concept could not be defined without resorting to circular reasoning except in the most unrealistic of models. Second, for any rate of interest there is a different short-period equilibrium in a neoclassical model. There are not enough equilibrium conditions to determine what this rate of interest is.

This paper focuses on the latter issue: whether the rate of interest is determinate in the neoclassical model. It is well known that this need not be true with overlapping generations: Kehoe and Levine (1985c) give a simple example of an overlapping generations model that has no cycles, is not chaotic, and has a robust continuum of Pareto-efficient equilibria that converge to the same Pareto-efficient stationary state. If we focus on the neoclassical case of the behavior of a production economy with a finite number of heterogeneous, infinitely lived consumers and equilibria that converge to a non-degenerate stationary state or cycle, however, we find that the set of equilibria are determinate, that is, locally unique for almost all endowments.

Our result shows that the determinacy of the rate of interest depends critically on whether or not there are finitely or infinitely many agents. The example in Kehoe and Levine (1985c), clearly shows that indeterminacy has nothing to do with whether or not equilibrium prices lie in the dual of the

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commodity space. More strongly, Kehoe, Levine, Mas-Colell, and Zame (1986) show that robust indeterminacy can arise when both the commodity and price spaces are the same Hilbert space, provided there are infinitely many consumers. Consequently, it is the assumption of finitely many consumers that drives our results in this paper.

Our results extend those of Muller and Woodford (1986), who consider production economies with both finitely and infinitely lived agents. They show that there can be no indeterminacy if the infinitely lived agents are sufficiently large. Their results are local, however, and concern only equilibria that converge to a particular stationary state. We prove a global theorem: for a given starting capital stock, there are only finitely many equilibria that converge to any non-degenerate stationary state. One particular implication is that when the discount factor is sufficiently close to one, which implies that there is a global turnpike, then equilibria are determinate.

We assume that markets are complete and that the technology and preferences are convex. Consequently, the behavior of equilibria in our model can be characterized by the properties of a value function. This is because the second theorem of welfare economics holds; that is, any Pareto-efficient allocation can be decentralized as a competitive equilibrium with transfer payments. If the preferences of consumers can be represented by concave utility functions, then an equilibrium with transfers can be calculated by maximizing a weighted sum of the individual utility functions subject to the feasibility constraints implied by the aggregate technology and the initial endowments. Showing that an equilibrium exists is equivalent to showing that there exists a vector of welfare weights such that the transfer payments needed to decentralize the resulting Pareto-efficient allocation are zero. This approach has been pioneered by Negishi (1960) and applied to dynamic models in a series of papers by Bewley (1980, 1982). Using this approach, Kehoe and Levine (1985a) have considered the regularity properties of an infinite horizon economy without production.

In general, calculating the transfers associated with a given set of weights requires the complete calculation of equilibrium quantities and prices. In a dynamic model with an infinite number of commodities, this can be awkward. To simplify the calculation, we adopt an alternative strategy based on the simple geometric observation that any convex set in $\mathbb{R}^n$ can be interpreted as the cross-section of a cone in $\mathbb{R}^{n+1}$. To exploit this fact, we add a set of artificial fixed factors to the economy and include them as arguments of the weighted social value function. These factors are chosen so that the augmented utility and production functions are homogeneous of degree one. Thus, the usual problem of choosing a point on the frontier of a convex utility possibility set is converted into a problem of choosing a point from a cone of feasible values for utility. This extension has theoretical advantages analogous to those that arise when a strictly concave production function is converted into a homogeneous of degree one function by the addition of a fixed factor. When the technology for the firm is a cone, profits and revenues are completely accounted for by factor payments. Analogously, making the social value function homogeneous of degree one simplifies the accounting necessary to keep track of the transfers associated with any given Pareto-efficient allocation. The present value of income and expenditure for each individual can be calculated directly from an augmented list of endowments and from the derivatives of the augmented social value function, without explicitly calculating the dynamic paths for prices or quantities. This is the framework for studying multi-agent intertemporal equilibrium models developed by Kehoe and Levine (1985b).

In such a setting equilibria are equivalent to zeros of a simple finite dimensional system of equations involving the derivatives of the social value function and the endowments. Intuition says that since the number of equations and the number of unknowns in this system are both equal to the number of agents, equilibria ought to be determinate. To do the usual kind of regularity analysis, however, we require that the system of equations that determines the equilibria be continuously differentiable. Because these equations involve derivatives of the social value function, they are C^1 if the value function is C^2.

Unfortunately, the question of when the value function is C^2 has not been entirely answered. Consequently, we are led to augment the system of equations with vectors of capital sequences, and focus on equilibria that converge to non-degenerate steady states. Using methods pioneered by Araujo and Scheinkman (1977), we can then prove determinacy using infinite dimensional transversality theory.

If the dimension of the stable or unstable manifold of a stationary state changes as the welfare weights change, or if the total number of stationary states changes, then the system bifurcates. In this case, the dynamical system at stationary states must have unit roots, and our theorem does not apply. Alternatively, the system may have cycles of unbounded length, in which case we loosely refer to it as chaotic. Consequently, our results may be summarized by saying that, in the class of economies that have no bifurcations and no chaos, determinacy is generic.

These results complement those of Kehoe, Levine, and Romer (1987), who make use directly of the differentiability of the value function. There it is shown that if the discount factor is large, there is a global turnpike and the value function is C^2. This overlaps with the results here. On the other hand, if the discount factor is small, the value function is also C^2. This covers most known examples of chaotic systems, as well as systems that satisfy the non-bifurcation condition outlined above. This paper shows that with the non-bifurcation condition, we do not need for the value function to be globally C^2, nor do we need discount factor conditions.

In the next section we present a highly aggregated neoclassical general equilibrium model. We use this model to motivate the approach that we follow and to provide an overview of our results. In Section 3 we establish some preliminary mathematical details about concave functions. Section 4 describes the economy. Section 5 defines and analyzes the savings function of the economy. In the Appendix we prove our major results on the genericity of regularity.

2 AN EXAMPLE

Consider a simple two-person neoclassical growth model. The preferences of each consumer take the usual additively separable form, discounted by the common factor β, $0<\beta<1$. The utility function for consumer i, $i=1,2$, is $\Sigma_{t=0}^{\infty}\beta^t u_i(c_{it})$ where the momentary utility function u_i is strictly concave and monotonically increasing. The initial endowment of the single productive factor is $\bar{k}^0$, and θ_i is the share of consumer i. Obviously, $\theta_1+\theta_2=1$, and $\theta_i\bar{k}_0$ is the endowment of consumer i. The technology is described by a strictly concave, monotonically increasing aggregate production function $g\colon R_+\to R$. Any profits are distributed to consumers in shares φ_i where $\varphi_1+\varphi_2=1$.

A competitive equilibrium for this model consists of a sequence $p_0,p_1,\ldots$ of prices for the consumption good, a price r for the initial capital stock, a consumption allocation $c_{i0},c_{i1},\ldots$ for each consumer i, a sequence of capital stocks $k_0,k_1,\ldots$, a sequence of outputs of the consumption good $q_0,q_1,\ldots$, and a level of total of profits π. Given the prices p_t and r_t and the profits π, the consumption allocation c_{it} must solve the utility maximization problem for consumer i:

$$\max \sum_{t=0}^{\infty} \beta^t u_i(c_{it})$$

$$\text{s.t.} \sum_{t=0}^{\infty} p_t c_{it} \leqslant \theta_i r\bar{k}_0+\varphi_i\pi.$$

Furthermore, given the prices p_t, the production plans k_t,q_t must maximize profits:

$$\max \sum_{t=0}^{\infty} p_t q_t - rk_0$$

$$\text{s.t. } q_t+k_{t+1}\leqslant g(k_t), \quad t=0,1,\ldots$$

In addition, the profits π that enters into the consumers' budget constraints must be those actually generated by the production plans k_t,q_t:

$$\pi = \sum_{t=0}^{\infty} p_t q_t - rk_0.$$

Finally, demand must equal supply for the consumption good in every period and for the initial capital stock:

$$c_{1t} + c_{2t} = q_t, \quad t = 0,1,\ldots$$

$$k_0 = \bar{k}_0.$$

Let us assume that the functions u_i and f are such that at any equilibrium the utility maximization problems of the consumers and the profit maximization problem of the production sector have interior solutions. Assuming that these functions are also continuously differentiable, we can characterize these solutions using first-order conditions. For the utility maximization problem of consumer i, these are

$$\beta^t u_i'(c_{it}) - \lambda_i p_t = 0, \quad t = 0,1,\ldots$$

$$\sum_{t=0}^{\infty} p_t c_{it} = \theta_i r \bar{k}_0 + \varphi_i \pi.$$

Here $\lambda_i > 0$ is the marginal utility of income of consumer i. The first-order conditions for profit maximization are

$$p_t - \mu_t = 0, \quad t = 0,1,\ldots$$

$$-r + \mu_0 g'(k_0) = 0$$

$$-\mu_{t-1} + \mu_t g'(k_t) = 0, \quad t = 1,2,\ldots$$

Here $\mu_t > 0$ is the Lagrange multiplier associated with the constraint in period t. These conditions can be simplified to

$$g'(k_0) = \frac{r}{p_0}$$

$$g'(k_t) = \frac{p_{t-1}}{p_t}, \qquad t = 1,2,\ldots$$

Let us now consider the social planning problem of determining a Pareto-efficient consumption allocation and production sequence. Given non-negative welfare weights (α_1,α_2), we maximize a weighted sum of the individual consumers' utilities subject to feasibility constraints:

$$\max \sum_{i=1}^{2} \alpha_i \sum_{t=0}^{\infty} \beta^t u_i(c_{it})$$

$$\text{s.t.} \sum_{i=1}^{2} c_{it} + k_{t+1} \leqslant g(k_t) \quad t=0,1,\dots$$

A solution to this problem can be characterized using the first-order conditions

$$\alpha_i \beta^t u_i'(c_{it}) - p_t = 0, \quad i=1,2;\ t=0,1,\dots$$

$$-p_{t-1} + p_t g'(k_t) = 0, \quad t=0,1,\dots$$

$$p_0 g'(k_0) - r = 0.$$

Here $p_t > 0$ is the Lagrange multiplier associated with the constraint on output in every period and r is the Lagrange multiplier associated with the constraint on the initial capital stock. (In addition to these conditions there is a transversality condition of the form $p_t k_t \to 0$ as $t \to \infty$.)

Notice that, if we set $\alpha_i = 1/\lambda_i$, $i=1,2$, then a competitive equilibrium satisfies all of the conditions for a Pareto-efficient consumption allocation and production sequence. (It is trivial to show that the transversality condition must be satisfied if the profit maximization problem has a finite solution.) This is the first theorem of welfare economics, that every competitive equilibrium is Pareto efficient. Notice too that, if we set $\lambda_i = 1/\alpha_i$, $i=1,2$, then a solution to the social planning problem satisfies all of the conditions for a competitive equilibrium except, possibly, the individual budget constraints. This is the second theorem of welfare economics, that every Pareto-efficient consumption allocation and production plan can be implemented as a competitive equilibrium with transfer payments. In this case, the transfer payments necessary to implement the consumption allocation and production plan associated with the welfare weights (α_1, α_2) are

$$\sum_{t=0}^{\infty} p_t(\alpha) c_{it}(\alpha) - \theta_i r(\alpha) \bar{k}_0 - \varphi_i \pi(\alpha), \quad i=1,2,\dots$$

where

$$\pi(\alpha) = \sum_{t=0}^{\infty} p_t(\alpha) \sum_{i=1}^{2} c_{it}(\alpha) - r(\alpha) \bar{k}_0.$$

A competitive equilibrium corresponds to a vector of welfare weights α for which these transfer payments are equal to zero.

Let us develop a characterization of solutions to the social planning problem, and of competitive equilibria, in dynamic programming terms. Given an aggregate endowment of capital k_0 and a vector of non-negative welfare weights (α_1,α_2), define a value function $V(k_0,\alpha_1,\alpha_2)$ as the maximum of

$$\sum_{i=1}^{2} \alpha_i \sum_{t=0}^{\infty} \beta^t u_i(c_{it})$$

subject to the constraint

$$\sum_{i=1}^{2} c_{it} + k_{t+1} \leqslant g(k_t).$$

The envelope theorem allows us to treat the derivative $D_1V(k_0,\alpha_1,\alpha_2)$ as the price for capital r and use it to calculate the value of the endowment $\theta_i k_0$ for each individual. To calculate the transfers associated with these weights, we must also calculate the profits of the firm, if any, and the expenditure of each individual. For profits this is straightforward: If f is not homogeneous of degree one, introduce a fixed factor $x \in R$ and define $G(k,x)=g(k/x)$. Specify endowments φ_i of this fixed factor equal to the ownership shares of the individuals in the aggregate firm. Then define $V(k_0,x,\alpha_1,\alpha_2)$ as the maximum of the weighted objective function subject to the constraint $\Sigma_{i=1}^{2} c_{it} + k_{t+1} \leqslant G(k_t,x)$. In equilibrium the aggregate endowment of the factor x is equal to 1, but it is useful to allow for the hypothetical possibility that it takes on other values so that we can calculate derivatives. Formally, we can treat x as a factor of production analogous to k and conclude that the share of the profits for agent i is φ_i multiplied by the price $D_2V(k_0,x,\alpha_1,\alpha_2)$. Since $G(k,x)$ is homogeneous of degree one, profits net of the new factor payments are zero. McKenzie (1959) has observed that any convex production possibility set could be interpreted as a cross-section of a cone in precisely this fashion and suggested that x be interpreted as an entrepreneurial factor.

Alternatively, x could denote the input of inelastic labor. In this interpretation each consumer is endowed with a constant amount φ_i of labor in every period, and units are normalized so that the total supply of labor in every period is one. What we have called profits is actually labor income. Here $g(k_t)=G(k_t,1)$ and our construction helps us recover the 'lost' factor.

The next step is to show that strictly concave utility functions can also be made homogeneous of degree one. If we interpret the fixed factor x as an accounting device used to keep track of producer surplus – the difference between revenue and expenditure – it is clear that a similar factor can be used to account for consumer surplus – the difference between utility and expenditure. Introduce an additional, person-specific fixed utility factor w_i

for each agent, and endow agent i with the entire aggregate supply of one unit of factor i. (For simplicity, we make no distinction in the notation between the individual's holdings of factor w_i and the aggregate endowment.) Just as we do for production, define an augmented utility function $U_i(c,w_i) = w_i u_i(c/w_i)$. In the next section, we show that this augmented utility function can always be defined and is well behaved even when u_i is unbounded from below. Now define a value function $V(k_0,x,w_1,w_2,\alpha_1,\alpha_2)$ as the maximum of the weighted sum of the augmented utility functions subject to the augmented technology.

If we let c_{it} denote the optimal consumption of agent i at time t, the first-order conditions from the maximization imply the equality

$$\beta^t \alpha_i D_1 U_i(c_{it},w_i) = \beta^t \alpha_j D_1 U_j(c_{jt},w_j).$$

As a result, discounted marginal utility for either consumer can be used as a present value price for consumption at time t. The only difference from the usual representative consumer framework is that the weights α convert the individual marginal utility prices into a social marginal value price. We can then evaluate the expenditure of consumer i in period t as c_{it} multiplied by this price. Using the properties of homogeneous functions, we can decompose period t utility for consumer i into the sum of a term of this form and an analogous term involving the added utility factor:

$$U_i(c_{it},w_i) = c_{it} D_1 U_i(c_{it},w_i) + w_i D_2 U_i(c_{it},w_i).$$

If the term involving the utility factor is interpreted as a measure of consumer surplus, expenditure on goods in period t is simply utility minus consumer surplus. Using the envelope theorem, we can calculate the present value of consumer surplus for agent 1 as the derivative of the social value function $V(k_0,x,w_1,w_2,\alpha)$ with respect to w_1 multiplied by the endowment w_1:

$$w_1 D_3 V(k_0,x,w_1,w_2,\alpha_1,\alpha_2) = \sum_{t=0}^{\infty} \beta^t \alpha_1 w_1 D_2 U_1(c_{1t},w_1).$$

Similarly, we can calculate the discounted sum of utility for consumer 1, measured in social value units, as the derivative of the social value function with respect to α_1 multiplied by α_1:

$$\alpha_1 D_5 V(k_0,x,w_1,w_2,\alpha_1,\alpha_2) = \sum_{t=0}^{\infty} \beta^t \alpha_1 U_1(c_{1t},w_1).$$

Then the present value of expenditure by agent 1 is simply the difference

$$\alpha_1 D_5 V(k_0,x,w_1,w_2,\alpha_1,\alpha_2) - w_1 D_3 V(k_0,x,w_1,w_2,\alpha_1,\alpha_2).$$

The transfer to agent 1 necessary to support this equilibrium is zero if and only if this expenditure is equal to the time zero value of the agent's endowment

$$\theta_1 k_0 D_1 V(k_0,x,w_1,w_2,\alpha_1,\alpha_2)+\varphi_1 D_2 V(k_0,x,w_1,w_2,\alpha_1,\alpha_2).$$

Formally, this equality can be interpreted in terms of an augmented economy where trade in the production factor x and the utility factors w_i actually takes place. In this case, this equality can be interpreted as a requirement that the value of the augmented endowment for agent 1, $\theta_1 k_0 D_1 V+\varphi_1 D_2 V+w_1 D_3 V$, equals the amount of social utility purchased, $\alpha_1 D_5 V=\alpha_1 \Sigma_{t=0}^{\infty} \beta^t U_1$.

It is useful to define a net savings function s_1 for consumer 1 as

$$s_1(k_0,\theta,\varphi,\alpha)=\theta_1 k_0 D_1 V(k_0,1,1,1,\alpha_1,\alpha_2)+\varphi_1 D_2 V(k_0,1,1,1,\alpha_1,\alpha_2)$$
$$+D_3 V(k_0,1,1,1,\alpha_1,\alpha_2)-\alpha_1 D_5 V(k_0,1,1,1,\alpha_1,\alpha_2).$$

The savings function for consumer 2 is defined symmetrically. For a given set of welfare weights α the transfer for each individual needed to support the social optimum as a competitive equilibrium is the negative of the net savings for that individual. A competitive equilibrium is therefore a vector of weights α such that $s(k_0,\theta,\varphi,\alpha)=0$. In general, if m is the number of individuals, $s(k_0,\theta,\varphi,\cdot)$ is a map from $\mathbb{R}^m$ into $\mathbb{R}^m$, and the existence of an equilibrium can be established using a standard fixed point argument in a finite dimensional space.

This characterization of equilibria as zeros of an equation involving endowments and the derivatives of an augmented value function is quite general. All that is required is that the second welfare theorem hold and that the preferences of the consumers can be represented using concave utility functions.

3 HOMOGENEOUS EXTENSION OF CONCAVE FUNCTIONS

In describing equilibria, we shall need the fact that it is possible to convert any concave utility function into a homogeneous function by adding a fixed factor. This does not follow immediately from results for production functions because utility functions need not be bounded from below. The analysis that follows would be considerably simpler if we restricted attention to utility functions that are bounded, but functions like logarithmic utility and isoelastic utility, $u(c)=-c^{-\sigma}$ where $\sigma>0$ are widely used in applications of this kind of model. In the formal analysis we accommodate these functions using the concepts and terminology from convex analysis for dealing with extended real valued functions; see Rockafellar (1970) for a complete treatment.

If n_c denotes the number of consumption goods in this economy, a utility function u is a function that is defined on the non-negative orthant $\mathbb{R}_+^{n_c}$ in commodity space, and that takes on values in $\mathbb{R}\cup\{-\infty\}$. On the strictly positive orthant $\mathbb{R}_{++}^{n_c}$, u is finite, but to accommodate functions like logarithmic utility, we want to allow for the possibility that $u(c)$ is equal to $-\infty$ if one of the components of c is equal to zero. We can define a topology on $\mathbb{R}\cup\{-\infty\}$ by adding the open intervals $[-\infty,a)$ to the base for the usual topology of $\mathbb{R}$. Note that $(-\infty,\infty)$ is not a closed set in this topology and that convergence to $-\infty$ has the usual interpretation: a sequence $\{b^n\}$ in $\mathbb{R}$ converges to $-\infty$ if, for all $M\in\mathbb{R}$, there exists an N such that $n\geqslant N$ implies $b^n\in[-\infty,M)$. With this topology, the natural assumption on preferences is that u: $\mathbb{R}_+^{n_c}\rightarrow\mathbb{R}\cup\{-\infty\}$ be continuous. For example, the utility functions $u(c)=-c^{-1/2}$ and $u(c)=ln(c)$ can both be represented as continuous functions from $\mathbb{R}_+$ to $\mathbb{R}\cup\{-\infty\}$.

The extension of $u(c)$ to the homogeneous function $U(c,w)=wu(c/w)$ does not preserve continuity on the non-negative orthant in $\mathbb{R}^{n_c+1}$. A discontinuity can arise at the point $(c,w)=(0,0)$. This extension does, however, preserve a weaker notion of continuity. Recall that for a function g: $\mathbb{R}^l\rightarrow\mathbb{R}$, g is ***upper-semi-continuous*** (u.s.c.) if the inverse image $g^{-1}([a,\infty))$ is always a closed set. If we allow the function g to take values in $\mathbb{R}\cup\{-\infty\}$ instead of $\mathbb{R}$, we can make an identical definition. Equivalently, g is u.s.c. if, for any sequence $\{y^n\}$ in $\mathbb{R}^l$ converging to y, $\lim_{n\rightarrow\infty}\sup g(y^n)\leqslant g(y)$. Since an u.s.c. function has a maximum over a compact set, upper-semi-continuity is strong enough for our purposes. If the function g is concave, define the ***recession function*** of g, r_g: $\mathbb{R}_+^l\rightarrow\mathbb{R}\cup\{-\infty\}$, as

$$r_g(y)=\lim_{t\rightarrow\infty}\frac{g(z+ty)-g(z)}{t}$$

where z is any point such that $g(z)$ is finite. Since g is concave, it can be shown that r_g is homogeneous of degree one and does not depend on the choice of z in the definition. Roughly speaking, $r_g(y)$ describes the asymptotic average slope of g along a ray from the origin passing through the point y.

Given these definitions, we can now state the key lemma for our construction. See Rockafellar (1970, p.67) for a proof.

Lemma 1: Let g: $\mathbb{R}_+^l\rightarrow\mathbb{R}\cup\{-\infty\}$ be concave and continuous. Let

$$G: \mathbb{R}^l\times\mathbb{R}\rightarrow\mathbb{R}\cup\{-\infty\}$$

be defined by

$$G(y,\rho) = \begin{cases} \rho g(y/\rho) & \text{if } \rho > 0 \text{ and } y \in R^{\ell}_{+}, \\ r_g(y) & \text{if } \rho = 0 \text{ and } y \in R^{\ell}_{+} \\ -\infty & \text{otherwise} \end{cases}$$

Then G is concave, u.s.c., and homogeneous of degree one.

If g is a production function, hence non-negative, $G(y,\rho)$ is increasing in ρ. If g represents a utility function that takes on negative values, $G(y,\rho)$ is decreasing in ρ for some values of y. In the artificial equilibrium where we allow for trade in the utility factors, this may imply that the price associated with the utility factor is negative. Implicitly, the strategy here is to consider first an equilibrium with explicit markets in all goods, including the fixed factors in the utility functions. Prices are such that each individual consumes his endowment (equal to one) of the utility factor. Then prices and quantities for all other goods do not depend on whether or not trade in the utility factors is possible. The possibility of negative prices for utility factors in the complete markets equilibrium poses no problem for proving existence because it is not necessary to assume free disposal of the utility factors. As long as each individual is endowed at time zero with a positive amount of capital or some other factor with positive value, strictly positive consumption of all true consumption goods is feasible. The total value of any individual's endowment may be negative, but it is always possible to use up the utility factor, that is, consume it, leaving strictly positive income to be spent on the true consumption goods.

4 FORMAL EQUILIBRIUM MODEL

Assume that there are m consumers in the economy. Let n_k denote the number of reproducible capital stocks, n_c the number of consumption goods. Let k_0 denote the n_k vector of initial aggregate capital stocks. Let φ denote the m vector of ownership shares for the single aggregate firm. Each agent has a utility function u_i: $R^{n_c} \to R \cup \{-\infty\}$; let U_i: $R^{n_c} \times R \to \cup R \times \{-\infty\}$ denote the homogeneous extension of u_i as defined in Lemma 1. Naturally $U_i(c_{it}, 1) = u_i(c_{it})$.

We assume that all consumers have the same discount factor $\beta > 1$. Conceptually, there is no difficulty with different consumers having different discount factors. Kehoe and Levine (1985b) show how to integrate this into the formal model. Moreover, the proof of the existence of an equilibrium remains straightforward. For simplicity, assume that the technology that relates period t to period $t+1$ can be described in terms of an aggregate production function. Let c_t, k_t, and k_{t+1} denote the aggregate consumption at time t, and capital at time t and $t+1$. Then the technology is described by the constraint $f(k_t,k_{t+1},c_t)\geq 0$, where f: $\mathbb{R}_+^{n_k}\times\mathbb{R}^{n_k}\times\mathbb{R}_+^{n_c}\rightarrow\mathbb{R}$. In our simple example,

$$f(k_t,k_{t+1},c_t)=g(k_t)-k_{t+1}-c_t.$$

Formally, it is convenient to allow f to be defined when the terminal stock k_{t+1} lies outside the non-negative orthant. Hypothetically, if it were possible to leave negative capital for next period, f describes the additional current consumption that would be possible. In the intertemporal optimization problems we explicitly impose the constraint that k_{t+1} be non-negative.

In this specification of the aggregate technology, we have not made explicit the dependence of output on factors of production that are in fixed supply. Formally, it is as if we have given ownership of all such factors to the aggregate firm. Individuals sell any endowments of land and labor for an increased ownership share in the firm. To consume a specified amount of leisure or of consumption services from land, an individual must purchase these like any other consumption good. This is merely a notational convenience. To make these factors explicit, we would simply need to augment the argument list for the production function and specify individual endowments in these additional factors.

By Lemma 1, there exists a homogeneous function $F(k_t,k_{t+1},c_t,x)$ such that $F(k_t,k_{t+1},c_t,1)=f(k_t,k_{t+1},c_t)$. Given the additional fixed factor x, the aggregate technology set is a cone. Its representation in terms of an aggregate production function is convenient because it allows a simple specification of the smoothness properties of the technology. If F is smooth, the boundary of the cone is smooth. A more general treatment along the lines of Bewley (1982) would start from assumptions about the separate technologies available to individual firms, but our interest here lies not so much with the specification of the technology, but rather with the specification of preferences and endowments.

We can now specify the properties assumed for the preferences and technology. The assumptions concerning continuity and smoothness are standard. For convenience, the usual monotonicity assumptions are strengthened, but this is not essential. A more important restriction is that the utility function be strictly concave and that the production possibility set for output capital stocks be strictly convex when the input capital stocks and aggregate consumption are held constant.

Assumption 1: For all i, the utility function u_i: $\mathbb{R}_+^{n_c}\rightarrow\mathbb{R}\cup\{-\infty\}$ is concave, strictly increasing, and continuous. On the strictly positive orthant $\mathbb{R}_{++}^{n_c}$, u is C^2 and has a negative definition Hessian.

Assumption 2: The production function f: $\mathbb{R}_+^{n_k} \times \mathbb{R}^{n_k} \times \mathbb{R}_+^{n_c} \to \mathbb{R}$ is concave and continuous, with $f(0,0,0)=0$). On the interior of its domain, f is C^2, strictly increasing in its first argument, and strictly decreasing in the second and third arguments. Also the matrix of second derivatives with respect to the vector of terminal stocks, $D_{22}f(k_t,k_{t+1},c_t)$, is negative definite.

In his discussion of the von Neumann facet, McKenzie (1983) has emphasized that it is restrictive to assume that f is strictly concave. If fixed factors in production can be allocated between different constant returns to scale industries (for example, labor in the multisector neoclassical growth model), there can exist an affine set of initial and terminal capital stocks that produce the same consumption goods vector. In the usual case where consumption and next period capital can be exchanged one for one, the weaker assumption that $D_{22}f(k_t,k_{t+1},c_t)$ is negative definite requires that, given k_t, the set of possible output combinations have a production possibility frontier with positive curvature.

If we define $f(k_t,k_{t+1},c_t)$ as $g(k_t)-k_{t+1}-c_t$ in our simple example, then $D_{22}f(k_t,k_{t+1},c_t)=0$. Suppose instead we set

$$f(k_t,k_{t+1},c_t)=h(g(k_t))-h(k_{t+1}+c_2)$$

where h is a function that satisfies $h'>0$ $h''>0$. If $g''<0$, then we can choose h so that the composition $h(g(k_t))$ is still concave. Consequently, f now satisfies Assumption 2. We should stress that we have made our assumptions very strong to keep our exposition as simple as possible. Most of our results could be derived under weaker assumptions.

Because some of the factors of production are in fixed supply, output exhibits diminishing returns as a function of the initial capital stock. The next assumption strengthens the diminishing returns so that feasible output stocks are bounded.

Assumption 3: (Boundedness) There exists $k_{max} \in \mathbb{R}_+^{n_k}$ and a bound $b<1$ such that $k_t \geqslant k_{max}$ and $k_{t+1} \geqslant bk_t$ implies that k_{t+1} is not feasible.

This assumption states that capital stocks larger than k_{max} cannot be sustained. By the definition of F, this bound also holds when $f(k_t,k_{t+1},c)$ is replaced by its homogeneous extension $F(k_t,k_{t+1},c,x)$ for any value of x less than or equal to one. This boundedness assumption is stronger than is needed for existence of a social optimum or an equilibrium, but it is required to rule out unbounded growth paths in the proof of the existence of an optimal stationary value for the capital stock.

Proving the existence of an optimal stationary state also requires the other half of a set of Inada-type conditions on production. Assumption 4 ensures that there exist strictly positive feasible paths for capital and consumption, and that at least one such path does not converge asymptotically to zero consumption and capital. Recall that $\mathbb{R}^k_{++}$ denotes the strictly positive orthant in $\mathbb{R}^k$ and that $\beta < 1$ is the discount factor.

Assumption 4: (Feasibility) For all $k_t \in \mathbb{R}^k_{++}$ there exists $k_{t+1} \in \mathbb{R}^k_{++}$ and $c \in \mathbb{R}^c_{++}$ such that (k_t, k_{t+1}, c) is feasible, that is, $f(k_t, k_{t+1}, c) \geqslant 0$. Furthermore, for some point $k_t \in \mathbb{R}^k_{++}$, k_{t+1} and c can be chosen so that $c \in \mathbb{R}^c_{++}$ and $\beta k_{t+1} > k_t$.

The smoothness arguments that follow require that the optimal values of the capital stock and consumption lie in the interior of the domain of the production function and the consumption function respectively. This is guaranteed here by infinite steepness conditions on the boundary of the domains. For a concave function g: $\mathbb{R}^r \to \mathbb{R} \cup \{-\infty\}$, the generalization of a derivative is a subgradient. The set of *subgradients* of g at y, denoted $\partial g(y)$, is defined by

$$\partial g(y) = \{p \in \mathbb{R}^r \colon g(z) - g(y) \leqslant p(z - y) \text{ for all } z \in \mathbb{R}^r\}.$$

Note that we follow the unfortunate, but well-established, convention of letting a term like subgradient have a different meaning for concave and convex functions. For a convex function $h(z)$, the definition of $\partial h(z)$ is given by reversing the direction in the inequality in the definition given here. Let $\{y^n\}$ be a sequence in $\mathbb{R}^r_{++}$. Suppose g is a differentiable function with the property that one of the components of the gradient $\partial g(y^n)$ has a limit equal to ∞ as y^n approaches a point y. Then $\partial g(y)$ is empty. By the assumption of concavity, a point like y can arise only on the boundary of the domain of g. For a function like $g(y_1, y_2) = y_1^{0.3} y_2^{0.3}$, the limit of the gradient as (y_1^n, y_2^n) goes to (0,0) cannot be defined, but it is still the case that $\partial g(0,0)$ is empty.

Assumption 5: (Infinite steepness on the boundary)

(a) If c is an element of the boundary of the domain of u_i, the set of subgradients $\partial u_i(c)$ is empty.

(b) If any component of k_t is 0, the set of subgradients of f with respect to its first argument, $\partial_1 f(k_t, k_{t+1}, c)$, is empty.

Part (a) implies that the marginal utility of any good is infinite starting from zero consumption of that good. As stated, it allows u to be finite or to equal $-\infty$ on the boundary. It implies that every individual consumes some amount of every good in equilibrium, but weaker conditions could be used. All that is necessary is that a strictly positive amount of each good be produced in equilibrium. Part (b) is the usual assumption of infinite marginal productivity of each capital good starting from zero usage.

5 SOCIAL RETURN AND SAVINGS FUNCTIONS

We now define the return and savings functions derived from the social planning problem. These are then used to define an equilibrium. Given the underlying preferences and technology, we define a weighted momentary social return function v: $\mathbb{R}^{nk}_{+} \times \mathbb{R}^{nk} \times \mathbb{R}_{+} \times \mathbb{R}^{m}_{+} \times \mathbb{R}^{m}_{+} \to \mathbb{R} \cup \{-\infty\}$ as follows: If $F(k_t,k_{t+1},0,x) \geqslant 0$, that is, if non-negative aggregate consumption is feasible,

$$v(k_t,k_{t+1},x,w,\alpha) = \max \sum_{i=1}^{\infty} \alpha_i U_i(c_i,w_i)$$

$$\text{s.t. } F\left(k_t,k_{t+1}, \sum_{i=1}^{\infty} c_{it},x \right) \geqslant 0$$

$$c_i \geqslant 0.$$

If $F(k_t,k_{t+1},0,x) < 0$,

$$v(k_t,k_{t+1},x,w,\alpha) = -\infty.$$

If we were to work only with utility functions that are bounded from below on a suitably chosen domain, v would be a familiar, real valued saddle function. It is concave and homogeneous of degree one in (k_t,k_{t+1},x,w), convex and homogeneous of degree one in α. It would also be continuous in the usual sense, instead of u.s.c. as established below. The following proposition, characterizing v, is proven in Kehoe, Levine, and Romer (1987).

Proposition 1: Under Assumptions 1–5, the following results hold:

(a) v is well defined.

(b) For $\alpha \in \mathbb{R}^{m}_{+}$, $v(\cdot,\cdot,\cdot,\cdot,\alpha)$ is concave, u.s.c., and homogeneous of degree one, with the same monotonicity properties as F.

(c) For any $(k_t,k_{t+1},x,w) \in \mathbb{R}^{nk}_{+} \times \mathbb{R}^{nk} \times \mathbb{R}_{+} \times \mathbb{R}^{m}_{+}$, the function $v(k_t,k_{t+1},x,w,\cdot)$: $\mathbb{R}^{m}_{++} \to \mathbb{R} \cup \{-\infty\}$ is convex and homogeneous of degree one.

(d) For any $(x,w,\alpha) \in \mathbb{R}_{+} \times \mathbb{R}^{m}_{+} \times \mathbb{R}^{m}_{+}$, the set of subgradients of the concave function $v(\cdot,\cdot,x,w,\alpha)$ is empty at every point on the boundary of its domain.

(e) On the interior of its domain, v is C^2.

(f) Evaluated at any point in the interior of the domain of v, $D_{22}v(k_t,k_{t+1},x,\alpha)$ is negative definite.

Next, we consider the optimization problem

$$\max \sum_{t=0}^{\infty} \beta^t v(k_t,k_{t+1},x,w,\alpha). \tag{1}$$

Here, the maximization is over all non-negative sequences $\{k_t\}$ having an initial value equal to k_0. The constraint that the sequence be feasible is implicit in the maximization problem since v must take on the value $-\infty$ at some point along an infeasible sequence. Let $\ell_\infty^{n_k}$ denote the Banach space of bounded sequences in $\mathbb{R}^{n_k}$ under the sup norm, $|k| = \sup_t |k_t|$, where $|k_t|$ denotes any norm equivalent to the usual norm on $\mathbb{R}^{n_k}$; let $(\ell_\infty^{n_k})_+$ denote the positive orthant in $\ell_\infty^{n_k}$. For convenience, assume that the first component of any sequence in $\ell_\infty^{n_k}$ has an index $t = 1$. Define the mapping associated with the Euler equation, ξ: $(\ell_\infty^{n_k})_{++} \times \mathbb{R}_{++}^{n_k} \times \mathbb{R}_{++}^{m} \rightarrow \ell_\infty^{n_k}$, by the rule

$$\xi(k,k_0,\alpha)_t = D_2 v(k_{t-1},k_t,1,1,\alpha) + \beta D_1 v(k_t,k_{t+1},1,1,\alpha), \quad t \geqslant 1.$$

By the usual sufficient conditions for concave maximization problems, any path k_t that remains bounded and satisfies the Euler equation $\xi(k,k_0,\alpha) = 0$ is an optimal path. Conversely, any optimal path k starting at an interior point $k_0 \in \mathbb{R}_{++}^{n_k}$ satisfies this equation and remains bounded. Consequently, $k \in (\ell_\infty^{n_k})_+$ is optimal if and only if $\xi(k,k_0,\alpha) = 0$.

To define the savings functions we need to specify the matrices of individual endowments. Let θ denote the $n_k \times m$ matrix of non-negative capital shares. Naturally $\Sigma_{j=1}^{m} \theta_{ij} = 1$. Let k_0 denote the $n_k \times n_k$ diagonal matrix of capital stock corresponding to an n_k vector k_0, and let A denote the $m \times m$ diagonal matrix of welfare weights corresponding to an m vector α. Also let φ denote the m vector of endowment shares of the fixed factor of production. We say that the endowment shares θ and φ and initial stock k_0 are *admissible* if all of the components are non-negative, if the aggregate supplies k_0, are strictly positive, if every individual is endowed with a positive amount of some capital good, and if the shares sum to one. If we let θ^T denote the transpose of θ and interpret all the following products as matrix products, we can define the savings function for any admissible k_0,θ,φ and $\alpha \in \mathbb{R}_+^m$ as follows:

$$\sigma(k,k_0,\alpha,\theta,\varphi) = \theta^T k_0 D_1 v(k_0,k_1,1,1,\alpha) + \varphi \sum_{t=0}^{\infty} \beta^t D_3 v(k_t,k_{t+1},1,1,\alpha)$$

$$+ \sum_{t=0}^{\infty} \beta^t D_4 v(k_t,k_{t+1},1,1,\alpha) - A \sum_{t=0}^{\infty} \beta^t D_5 v(k_t,k_{t+1},1,1,\alpha).$$

Notice that, in defining the savings function, we have set $x = 1$ and $w_i = 1$ for $i = 1,\ldots,m$. At these values the augmented functions $U_i(c_{it},w_i)$ and $F(k_t,k_{t+1},c_t,x)$ reduce to the original specifications u_i and f. The next proposition establishes the basic properties of σ, and of the optimal stationary capital stock k^{ss}.

Proposition 2: Let $k_0 \in \mathbb{R}_+^{n_k}$, let $x \in \mathbb{R}_{++}$, and let $\alpha \in \mathbb{R}_+^m$. Under Assumptions 1–5 the following results hold:

(a) The maximization problem (1) has a unique solution.
(b) There exists an optimal stationary value $k^{ss}=k^{ss}(\alpha)$; that is, the sequence defined by $k_t=k^{ss}$ solves the problem (1) beginning at k^{ss}; every optimal stationary value lies strictly in the interior.
(c) The pair (ξ,σ) is C^1.
(d) The pair (ξ,σ) is homogeneous of degree one in α.
(e) $\Sigma_{i=1}^{n}\sigma_i(k,k_0,\alpha,\theta,\varphi)\equiv 0$.

See the Appendix for the proof. Kehoe, Levine, and Romer (1987) prove the following result.

Proposition 3: For given k_0, θ, and φ, k and α are an equilibrium if and only if

$$\xi(k,k_0,\alpha)=0$$

$$\sigma(k,k_0,\alpha,\theta,\varphi)=0.$$

Moreover, an equilibrium exists for any k_0, θ, and φ that are admissible.

Because (ξ,σ) is homogeneous and $\Sigma_{i=1}^{m}\sigma_i=0$, we should delete one variable and one equation from our equilibrium system. Fix $\alpha_m=1$. For notational simplicity we assume hereafter that $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$. We shall also assume hereafter that σ consists of σ_1 to σ_{m-1} only. Our procedure is analogous to that used with systems of excess demand functions where homogeneity of degree zero is used to impose a numeraire and Walras's Law is used to drop an equation.

We now define a non-degenerate steady state and prove that, for generic initial conditions, there are only finitely many equilibria converging to non-degenerate steady states. Define $D_h\xi=[D_1\xi\ D_2\xi]$. Let $D_{ij}v_t$ denote $D_{ij}v(k_t,k_{t+1},\alpha)$. Then we can write the component t of $D_h\xi(k,k_0,\alpha)h$ as

$$\beta D_{12}v_t h_{t+1}+(\beta D_{11}v_t+D_{22}v_{t-1})h_t+D_{21}v_{t-1}h_{t-1}.$$

In other words, $D_h\xi h=0$ gives rise to a linear dynamical system. At a stationary state, the coefficients are time independent, so we omit the time subscript. For emphasis we write $D_h\xi^{ss}$ to emphasize we are considering a stationary state. By the roots of $D_h\xi^{ss}$ we mean the eigenvalues of the associated linear dynamical system. We call a stationary state *non-degenerate* if $D_h\xi^{ss}$ has no roots on the unit circle and if $D_{12}v$ is non-singular.

That $D_{12}v$ is non-singular ensures that the dynamical system can be solved both forwards and backwards since $D_{21}v$ is the transpose of $D_{12}v$. In economic terms, this means that the model must be stated using a minimal set of capital goods. To see what this rules out, consider a Cobb–Douglas neoclassical growth model stated in terms of two capital goods, two consumption goods, and a fixed endowment of labor that must be allocated between identical production functions for the two consumption goods. Let two individuals have identical preferences $ln(c_1)+ln(c_2)$. The economy satisfies Assumptions 1–5. It is straightforward to show by direct algebraic manipulation that independent of the initial stocks of capitals, the subsequent aggregate stocks of the capital goods in the social planning problem for and set of weights α are always chosen in fixed proportions. (The consumption goods are also consumed in fixed proportions.) The model is not in any relevant sense two dimensional: the two capital goods and the two consumption goods can be combined into single composite capital and consumption goods. The dynamical system associated with the social planning problem for this economy always maps k_{t+1} onto a line in R^2. One can also show directly that $D_{12}v(k_t,k_{t+1},\alpha)$ is everywhere singular in this case. This kind of collapse in the dimensionality of the model is prevented, even locally, by assuming that $D_{12}v$ is non-singular.

At a non-degenerate steady state, it is well known that R^{2n_k} can be written as the direct sum of a stable and unstable manifold. We refer to n_k minus the dimension of the stable manifold as the *index* of $D_h\xi^{ss}$. In Lemma 3 below we show that $D_1\xi^{ss}$ is one to one. It follows directly that the index is non-negative. We call a path k non-degenerate for α and k_0 if k_t converges to a non-degenerate stationary state $k^{ss}(\alpha)$ and if, whenever index $k^{ss}(\alpha)\geqslant 1$, $D_{12}v(k_t,k_{t+1},\alpha)$ is non-singular for $t=0,1,\ldots$.

Note that these definitions may easily be extended to allow cycles in place of stationary states. Consider a cycle with period p. We redefine periods with n_cp commodities and n_kp types of capital per period so that all cycles appear as stationary states. This kind of trick is frequently used with overlapping generations models. Consequently, the subsequent propositions apply equally to paths converging to cycles.

Let $\bar{E}(k_0)$ denote the set of pairs $(k,\alpha)\in(\ell^{n_k}_\infty)_+\times R^{n-1}$ such that k is non-degenerate for α and k_0. In other words, we restrict attention to paths that converge to a non-degenerate stationary state. Our goal in this section is to prove the following result:

Proposition 4: For fixed φ and a full measure subset of k_0 and θ there are finitely many equilibria in $\bar{E}(k_0)$.

Notice incidentally, that by Fubini's Theorem, the fact that Proposition 4 holds for fixed φ and a full measure subset of k_0 and θ implies that it holds for a full measure subset of k_0, θ, and φ.

We can now define an equilibrium to be regular if the operator

$$\Sigma=\begin{bmatrix} D_1\xi & D_3\xi \\ D_1\sigma & D_3\sigma \end{bmatrix}$$

is non-singular. Since the inverse function theorem and implicit function theorem work as well in infinite dimensions as in finite, it follows that the equilibria of a regular economy are locally unique. Unfortunately, since $(\ell k) \times \mathbb{R}^{n-1}$ is infinite dimensional, the set of possible equilibrium values of (k, α) is not compact. We do not know, therefore, that the number of equilibria is necessarily finite.

Our preliminary goal is to study the circumstances under which Σ is non-singular.

Lemma 2: $D_1\xi$ is one to one.

Proof: Araujo and Scheinkman (1977) provide a proof under the assumption that $v(\cdot,\cdot,\alpha)$ is strictly concave, but this is stronger than is necessary. Proposition 2 demonstrates the uniqueness of solutions for this model and this is all that is required for their argument.

QED

With this preliminary we can now give a sufficient condition for Σ to be non-singular.

Lemma 3: If Σ is onto, then it is non-singular.

Proof: We must show Σ is one to one. Let

$$\kappa = \{h_\alpha \in \mathbb{R}^{n-1} | \text{ there exists } h_\kappa \in \ell_\infty^{n_k} \text{ such that } D_1\xi h_k + D_3\xi h_\alpha = 0\}.$$

Since $D_1\xi$ is one to one by Lemma 2, there is a unique linear operator B: $\kappa \to \ell_\infty^{n_k}$ such that $D_1\xi B h_\alpha + D_3\xi h_\alpha = 0$. Notice that, since B has finite dimensional domain, it is a continuous operator.

Suppose $h \in \ker \Sigma$. Then $h_\alpha \in \kappa$ and $h_k = Bh_\alpha$, which implies that $(D_1\sigma B + D_3\sigma)h_\alpha =)$. On the other hand, $D_1\sigma B + D_3\sigma$ is onto. Let $y_\alpha \in | \mathbb{R}^{n-1}$ and let $0 \in \ell_\infty^{n_k}$ with $y = (0, y_\alpha)$. Since Σ is onto, let $\bar{h}$ be a solution of $\Sigma \bar{h} = y$. Then $\bar{h}_\alpha \in \kappa$ and $\bar{h}_k = B\bar{h}_\alpha$. This implies that $(D_1\sigma B + D_3\sigma)\bar{h}_\alpha = y_\alpha$, which implies that $D_1\sigma B + D_3\sigma$ is onto. Finally, since $D_1\sigma B + D_3\sigma$ is a finite dimensional square matrix, it is also one to one. We conclude that if $h \in \ker \Sigma$, since $(D_1\sigma B + D_3\sigma)h_\alpha = 0$, then $h_\alpha = 0$. Since $h_k = Bh_\alpha$, we find that $h = 0$.

QED

We should emphasize that the picture is already very different from that with infinitely many agents. If we follow standard practice in infinite dimensional transversality theory, we would call an equilibrium regular if Σ is onto and its kernal has closed complement. We have just shown that this definition of regularity implies that Σ is non-singular. This should be contrasted with the robust indeterminacy that occurs with infinitely many agents. In that case the fact that Σ is regular, that is, onto, does not imply that it is one to one. The kernal of Σ is simply the tangent space to the manifold of equilibria. Since the manifold deforms smoothly with respect to small perturbations, they change neither the fact that Σ is regular, nor the dimension of the kernal. The indeterminacy is robust. For a more detailed discussion of this point, the reader is referred to Kehoe, Levine, Mas-Colell, and Zame (1986).

Lemma 4: At a non-degenerate steady state $D_h\xi^{ss}$ is onto and dim ker $D_h\xi^{ss} = n_k - \text{index } D_h\xi^{ss}$.

Proof: That dim ker $D_h\xi^{ss} = n_k - \text{index } D_h\xi^{ss}$ means that dim ker $D_h\xi^{ss}$ has the same dimension as the stable manifold; since multiple solutions to $D_h\xi^{ss}h = 0$ are indexed by pairs (h_0, h_1) on the stable manifold, this follows. That $D_h\xi^{ss}$ is onto follows from the fact that the stable manifold is robust at a non-degenerate stationary state with respect to small non-stationary perturbations; see the proof of the local stable manifold theorem in Irwin (1980). Consequently, $D_h\xi^{ss}h = b$ has non-empty stable manifold for small enough b, and since it is linear, for all b. In particular, $D_h\xi^{ss}h = b$ has at least one solution.

QED

The next task is to show that, if k converges to a non-degenerate stationary state, then $D_h\xi(k, k_0, a)$ is onto.

Proposition 5: If k is a non-degenerate path for α and k_0, then $D_h\xi(k, k_0, \alpha)$ is onto and has index equal to that at $k^{ss}(\alpha)$.

Proof: First we show that $D_h\xi$ is onto. Araujo and Scheinkman (1977) give a proof for the case where index $D\xi^{ss} = 0$. We examine only the case where index $D\xi^{ss} \geq 1$. Let $F\colon \ell_\infty^{n_k} \to \ell_\infty^{n_k}$ be defined by the rule $Fk = (k_2, k_3, \ldots)$. Since $k_t \to k^{ss}(\alpha)$ and small perturbations of $D_h\xi^{ss}$ are also onto, for some finite T, $D_h\xi(F^T k, k_T, \alpha)F^T h = F^T b$. Then, since $D_{12}v_t$ is by assumption non-singular, we simply solve recursively backwards to find

$$h_{t-1} = -D_{21}^{-1}v_{t-1}[(\beta D_{11}v_t + D_{22}v_{t-1})h_t + \beta D_{12}v_t h_{t+1} - b_t].$$

Since only a finite number of steps are involved, $h \in \ell_\infty^{n_k}$.

The fact that $D_h\xi$ and $D_h\xi^{ss}$ have the same index follows from the fact [shown, for example, in Araujo and Scheinkman (1977)] that they differ by a compact operator, and the fact that the index of a Fredholm operation is invariant under the addition of a compact operator.

QED

Let $E_i(k_0)$ denote the set of pairs $(k, \alpha) \in (\ell_\infty^{n_k})_+ \times R^{n-1}$ such that k is non-degenerate for α and k_0 and is of index i. Recall now that $0 \leq i \leq n_k$. We are interested in $E(k_0) = \cup_{i=0}^{n_k} E_i(k_0)$.

Proposition 6: For any fixed φ and a full measure subset of k_0 and θ every equilibrium in $\bar{E}(k_0)$ is regular.

Proof: In steps 1–4 we consider a fixed index i and $\bar{E}_i(k_0)$.

Step 1

We must find an open domain for ξ in order to do calculus. If $(k,\alpha)\in\bar{E}_i(k_0)$, then there is an open neighborhood $E_i(k_0)$ of (k,k_0,α) such that if $(k',k_0',\alpha')\in E_i(k_0)$ and $\xi(k',k_0',\alpha')=0$, then $(k',\alpha')\in\bar{E}_i(k_0')$; in other words, locally paths either converge to a non-degenerate stationary state, or leave; they do not remain bounded nearby without converging. This is shown in the proof of the robustness of the stable manifold; see, for example, Irwin (1980). We may also assume that, in $E_i(k_0)$, $D_h\xi(k',k_0',\alpha')$ is onto and has kernal of dimension n_k-i. This follows from the facts that the set of operators of this type is an open set (see Abraham and Robbin, 1967) and that $D_h\xi$ is a continuous function of its arguments by Proposition 2. Finally, let $E_i-\cup_{k_0}E_i(k_0)$. This open set we take to be the domain of ξ.

Step 2

Consider the matrix function on $E_\ell\times\mathbb{R}^{n_k(m-1)}$

$$\begin{array}{cc} & \begin{array}{cccc} \underline{k} & \underline{k_0} & \underline{\alpha} & \underline{\theta} \end{array} \\ \mu= & \begin{bmatrix} D_1\xi & D_2\xi & D_3\xi & 0 \\ D_1\sigma & D_2\sigma & D_3\sigma & D_4\sigma \end{bmatrix} \end{array}$$

In Kehoe, Levine, and Romer (1987) it is shown that $D_2\sigma$ is onto; and by construction $[D_1\xi\ D_4\xi]=D_h\xi$ is into. It follows that μ is onto. Moreover, since $D_4\sigma$ is non-singular, and $[D_1\xi\ D_2\xi]$ is onto with an n_k-i dimensional kernal, it is clear that $\dim\ker\mu=n_k-i+m-1$. The implicit function then implies that the set of (k,k_0,α,θ) such that (α,k) is an equilibrium is an $n_k-\ell+m-1$ dimensional C^1 manifold.

Step 3

To apply the parametric transversality theorem in step 4 below, we must show that the equilibrium manifold is second countable; that is, that every open covering has a countable subcovering. Since $\ell_\infty^{n_k}$ is not separable, it is not itself second countable. It is clearly sufficient, however, that the set of (k,k_0,α) in E_i with $\xi(k,k_0,\alpha)=0$ is second countable. By construction such k converge to a non-degenerate stationary state, and such convergence must be exponential, so it suffices to show that the space of convergent sequences converging at the rate $1/t$ is second countable. This is the product of the second countable space $\mathbb{R}^{n_k}$, containing the limits, and the space of sequences converging to zero at the rate $1/t$. The latter space of sequences is second countable because it is the union of sequences dominated by N/t as $N \to \infty$, and each of these spaces is compact. Finally, we observe that the product of second countable spaces is second countable.

Step 4

This step is identical to the finite dimensional proof of the parametric transversality theorem. See, for example, Abraham and Robbin (1967). Consider the projection $\Pi(k,k_0,\alpha,\theta)=(k_0,\theta)$ restricted to the equilibrium manifold. This is a C^1 map between second countable $n_k - i + (m-1)$ and $n_k + (m-1)$ dimensional C^1 manifolds; moreover, the point (k,k_0,α,θ) is a regular equilibrium if and only if it is a regular value of Π. By Sard's Theorem, however, the set of regular values (k_0,θ) are of full measure. This shows regular equilibria are full measure for each i.

Step 5

Since the countable union of measure zero sets has measure zero, the intersection of the full measure sets for each i has full measure.

QED

Observe that in step 4 the map to which Sard's Theorem applies is from an $n_k - i + m - 1$ dimensional manifold to an $n_k + m - 1$ dimensional one. It follows directly that if $i \geqslant 1$ then the equilibria in $E_i(k_0)$ are regular by virtue of not existing at all. Moreover, as we have remarked above, Araujo and Scheinkman show that $D_1\xi$ is onto at a steady state with index 0. Consequently, under the hypothesis of Proposition 4, we may assume that there are finitely many equilibria, and that every equilibrium has $D_1\xi$ non-singular.

APPENDIX

Proof of Proposition 2: See Kehoe, Levine, and Romer (1987) for the proof of parts (a), (b), (d), and (e).

To show part (c) consider any function $\psi_t = w(k_t,k_{t+1},\alpha)$ where w is C^1. Since (k_t,k_{t+1},α) may be restricted to a compact domain, the operator defined by

$$(D\psi(k,\alpha)h)_t = D_1w(k_t,k_{t+1}),\alpha)h_t + D_2w(k_t,k_{t+1},\alpha)h_{t+1} + D_3w(k_t,k_{t+1},\alpha)h_\alpha$$

is bounded and therefore continuous. Moreover, if $|k-k'|, |\alpha-\alpha'| \leqslant \varepsilon$ and $|h| \leqslant 1$, then

$$\sup |[D\psi(k,\alpha) - D\psi(k',\alpha')]h|$$

$$\leqslant 3 \sup |D_j w(k_t,k_{t+1},\alpha) - D_j w(k'_t,k'_{t+1},\alpha')|.$$

The compactness of the domain implies that $D_j w$ is uniformly continuous: as $\varepsilon \to 0$, $|D\psi(k,\alpha) - D\psi(k',\alpha')| \to 0$, in other words $D\psi$ varies continuously.

Finally, we can show that $D\psi$ is actually the derivative of ψ by again using the uniform continuity of $D_j w$ to show that the integral form of the remainder in period t, which is made of terms of the form

$$\int_0^1 (1-s)[D_j w(k_{t-1} + sh_{t-1}, k_t + sh_t, \alpha + sh_\alpha) - D_j w(k_{t-1}, k_t, \alpha)]ds$$

vanishes uniformly across periods as $h \to 0$.

This shows that ξ is C^1. Moreover, the mapping B: $\ell^{n_k}_\infty \to \mathbb{R}$ defined by $B(k) = \Sigma_{t=0}^{\infty} \beta^t k_t$ is continuous and linear, and thus C^∞. Since σ is then a composition of the form $B(\psi)$, it too is C^1.

QED

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