Continuous Time Models of Repeated Games with Imperfect Public Monitoring

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WUSTL Theory Workshop

What Happens In Repeated Games With Short Periods?

- A common model: continuous time limit
- Two effects in general: player more patient, information less good
- Impact of distribution of signals in a fixed discrete-time game
- Change in distribution with the period length
- Focus on case of long-run versus short-run
- Abreu, Pearce and Milgrom [1991], Sannikov [2006], Sannikov and Skrypcaz [2006], Faingold and Sannikov [2005], Faingold [2005]
- What is the underlying economics of all these results?

Basics – Long Run versus Long Run

Fudenberg, Levine and Maskin [1995] folk theorem

Under mild informational conditions any individually rational payoff vector approximated by equilibrium payoff if common discount factor of the players is sufficiently close to one

Sannikov [2005] characterizes equilibrium payoffs in continuous time where information follows vector valued diffusion, proves a folk theorem when information has a product structure and limit of interest rates $r \to 0$.

Basics – Long Run versus Short Run

Fudenberg and Levine [1994] LP algorithm to compute limit of equilibrium payoffs as discount factor of the long-run players converges to one and characterizes limit payoffs when information has a product structure; typically bounded away highest payoff when all players are long-run, but better than static Nash

Faingold and Sannikov [2005] show set of equilibria in continuous time where information is a diffusion process is only the static equilibrium

Abreu, Pearce and Milgrom [1991] implicitly show that with continous time Poisson information “bad news” signals lead to folk theorem, “good news” signal lead to static Nash

Summary

- Long run versus long run – length of period makes little difference
- Long run versus short run – length of period makes a big difference
  - “good news” Poisson or diffusion leads to static Nash
  - “bad news” Poisson leads to folk theorem

Long-Run versus Short-Run

two-person two-action stage game payoff matrix

	Player 2		
Player 1		**L**	**R**
	+1	$\underline{u}$,0	u,1
	-1	$\underline{u}$,0	$u+g$,-1

$\underline{u} < u, g > 0$

2 plays **L** in every Nash equilibrium

player 1's static Nash payoff $\underline{u}$, also minmax payoff

player 1 prefers that player 2 play **R**

can only induce player to play **R** by avoiding playing **–1**

classic time consistency problem

Information

end of stage game public signal $z \in \mathbb{R}$ observed

depends only on action taken by player 1

(player 2's action publicly observed)

public signal drawn from $F(z \mid a_1)$

F is either differentiable and strictly increasing

or corresponds to discrete random variable

$f(z \mid a_1)$ denotes density function

monotone likelihood ratio condition

$f(z \mid a_1 = -1) / f(z \mid a_1 = +1)$ strictly increasing in z

means that z is "bad news" about player 1's behavior in sense that it means player 1 probably playing **–1**

Other Stuff

Availability of public randomization device

τ length of period

player 1 long-run player with discount factor $\delta = 1 - r\tau$

player 2 an infinite sequence of short-run opponents

Best Perfect Public Equilibrium for LR

largest value v that satisfies incentive constraints

$$v = (1-\delta)u + \delta \int w(z) f(z \mid a_1 = +1) dz$$

$$v \geq (1-\delta)(u+g) + \delta \int w(z) f(z \mid a_1 = -1) dz$$

$$v \geq w(z) \geq \underline{u}$$

or $v = \underline{u}$ if no solution exists

second incentive constraint must hold with equality

otherwise increasing the punishment payoff w retains incentive compatibility and increases utility on the equilibrium path

Cut-Point Equilibria

monotone likelihood ratio condition implies these best equilibria have a cut-point property

$\tilde{z}^*$ is cut point

continuous z: a fixed cut-point

discrete z: a cut-point randomized between two adjacent grid-points

Proposition 1: There is a solution to the LP problem characterizing the most favorable perfect public equilibrium for the long-run player with the continuation payoffs $w(z)$ given by

$$w(z) = \begin{cases} w & z \geq \tilde{z}^* \\ v & z < \tilde{z}^* \end{cases}$$

and indeed, $w = \underline{u}$

Measures of Information

continuous case define

$$p = \int_{z^*}^{\infty} f(z \mid a_1 = +1)dz, q = \int_{z^*}^{\infty} f(z \mid a_1 = -1)dz$$

interested in case in which τ is small

information $q(\tau), p(\tau)$ functions of τ

$\rho, \mu \in \Re \cup \{\infty\}$ *regular values* of $q(\tau), p(\tau)$ if along some sequence $\tau^n \to 0$

$\rho = \lim_{\tau^n \to 0}(q(\tau^n) - p(\tau^n)) / p(\tau^n)$ [signal to noise]

$\mu = \lim_{\tau^n \to 0}(q(\tau^n) - p(\tau^n)) / \tau^n$ [signal arrival rate]

$v^* =$ [illegible] $\lim_{\tau^n \to 0} \max\{\underline{u}, v^*\}$

(***) $\mu\left(\frac{(u-\underline{u})}{g}-\frac{1}{\rho}\right)$

if positive and $\bar{v} > \underline{u}$ there is a *non-trivial* limit equilibrium

exists positive τ, r such that for all smaller values exists equilibrium giving long-run player more than $\underline{u}$

conversely, if either $\bar{v} \le \underline{u}$ or (***) is non-positive then for any fixed $r > 0$ along the sequence τ^n the best equilibrium for long-run converges to $\underline{u}$

if $\bar{v} = u$ say that limit equilibrium is efficient: if and only if

Proposition 2: Suppose that ρ, μ are regular. Then there is a non-trivial limit equilibrium if and only if $\rho > g/(u-\underline{u})$ *and* $\mu > 0$ and $\rho > 0$. There is an efficient limit equilibrium if and only if $\mu > 0$ and $\rho = \infty$.

Poisson Case

public signal of long-run generated by continuous time Poisson

Poisson arrival rate is

λ_p if action is **+1**

λ_q if action is **–1**

"good news" signal means probably played **+1**: $\lambda_q < \lambda_p$; z number of signals

"bad news" signal means probably played **–1**: $\lambda_q > \lambda_p$; z negative of number of signals

bad-news case $\lambda_q > \lambda_p$

cutoff number of signals before punishment $v - w$

two or more signals isn't interesting since probability of punishment is only of order τ^2

suffices to consider the cutoff in which punishment always occurs whenever any signal is received

probability of punishment $p(\tau) = 1 - e^{-\lambda_p \tau}, q(\tau) = 1 - e^{-\lambda_q \tau}$, as the long-run player plays **–1** or **+1**

then $\rho = (\lambda_q - \lambda_p)/\lambda_p, \mu = \lambda_q - \lambda_p$ (big and positive respectively)

$v^* = u - g\lambda_p / (\lambda_q - \lambda_p)$

note independence of payoff $\underline{u}$

good news" case $\lambda_q < \lambda_p$

punishment triggered by small number of signals, rather than large

if there is punishment, must occur when no signals arrive

probability of punishment when no signal $\gamma(\tau)$

$$p(\tau) = \gamma(\tau)e^{-\lambda_p \tau}, q(\tau) = \gamma(\tau)e^{-\lambda_q \tau}$$

regardless of $\gamma(\tau)$ implies $\rho = 0$, so only trivial limit

Overview

with short run providing incentives to long-run has non-trivial efficiency cost

"good news" case, providing incentives requires frequent punishment

many independent and non-trivial chances of a non-trivial punishment in a small interval of real time, long run player's present value must be low

contrast, can be non-trivial equilibrium even in the limit when signal used for punishment has negligible probability (as in bad-news case)

<u>or</u> several long run players so punishments can take the form of transfers payments

The Diffusion Case

signals generated by diffusion process in continuous time

drift controlled by the long-run action

sample process at intervals of length τ implies signals have variance $\sigma^2\tau$

we allow the variance signal $\sigma^2\tau^{2\alpha}$ where $\alpha < 1$, with diffusion corresponding to $\alpha = 1/2$

mean of the process is $-a_1\tau$ (recall that $a_1 =$**+1** or **–1**)

so:

$$p = \Phi\left(\frac{-z^* - \tau}{\sigma\tau^\alpha}\right)$$

$$q = \Phi\left(\frac{-z^* + \tau}{\sigma\tau^\alpha}\right)$$

where Φ is standard normal cumulative distribution

Proposition 3: For any $\alpha < 1$ there exists $\underline{\tau} > 0$ such that for $0 < \tau < \underline{\tau}$ there is no non-trivial limit equilibrium

true even when $\alpha > 1/2$, where process converges to deterministic one

contrast "bad news" Poisson case: like diffusion case corresponds to $\alpha = 1/2$

exact form of noise matters: is it a series of unlikely negative events, as in the "bad news" Poisson case, or a sum of small increments as in the normal case?

contrast the diffusion case $\alpha = 1/2$ with a sum of small increments where the scale of the increment is proportional to the length of the interval

standard error of the signal of order τ

corresponds to case $\alpha = 1$

take the limit of such sequence of processes, limit is deterministic process without noise.

Proposition 4: If $\alpha = 1$ there exists $\underline{\tau}$ such that for all $0 < \tau < \underline{\tau}$ (*) is satisfied, and $\lim_{\tau \to 0} v^* = u$.

for fixed τ taking a very large cutoff $z^* \to \infty$

causes the likelihood ratio $q/p \to \infty$, so $p/(q-p) \to 0$

so $v^* \to 1$

note $p,q \to 0$, so for fixed $\underline{u}$ and τ and z^* sufficiently large, (*) must be violated

for any choice of z^*, r, τ, there always $\underline{u}$ sufficiently small that (*) holds

worst punishment determines the best equilibrium

going far enough into tail of normal, arbitrarily reliable information can be found about whether a deviation occurred

information can be used to create incentives, provided sufficiently harsh punishment available

when $\alpha = 1$ signal to noise ratio improves sufficiently quickly that we can exploit the shorter intervals to choose a bigger cutoff value of ζ

Diffusion as a Limit

families of games indexed by period length τ

signal z varies with the period length τ

basic scenario: z an aggregate of discrete random variables

examples: sales, prices, or other transaction data

specifically z the sum of some number of "events" - independent identically distributed random variables Z_j with

support of Z_j a fixed finite set, regardless of action profile.

Wavelength and Frequency

time between moves, is τ

“observation frequency” $1/\tau$

length of time between events (realizations of Z_j) is $\Delta \leq \tau$

event frequency $1/\Delta$

assume τ an integer multiple of Δ

$k = \tau/\Delta$ events per period

case of interest: $\tau \to 0$ (forces $\Delta \to 0$)

assume τ a continuous strictly increasing function of Δ with $\tau(0) = 0$

distribution of Z_j and its support depend on Δ

cardinality of the support of Z_j is a constant, independent of Δ.

Converging to Diffusions: General Results

information available at end of period t

$$z = \sum_{j=t/\Delta}^{(t+\tau)/\Delta} Z_j$$

basic diffusion hypothesis: for each fixed action $i = +1, -1$ of long-run player

$z = \sum_{j=1}^{\lfloor t/\Delta \rfloor} Z_j$ converges to a diffusion as $\Delta \to 0$

simplest case $\Delta = \tau$

maximum possible value of q / p: punishing only on signal maximizing $f^{\Delta}(Z \mid -1) / f^{\Delta}(Z \mid +1)$

Define $M(k,\Delta) = \max_Z \left(f^{\Delta}(Z \mid -1) / f^{\Delta}(Z \mid +1) \right)^k$.

Proposition 6: *Suppose that* $\lim_{\Delta \to 0} \tau(\Delta) / \Delta = k < \infty$. *Then*

(a) *If* $\limsup_{\Delta \to 0} M(k,\Delta) < \infty$ *there is no efficient patient equilibrium.*

(b) *If* $\limsup_{\Delta \to 0} M(k,\Delta) = 1$ *there is only a trivial limit equilibrium.*

three cases

$\sigma_{+1}{}^2$ variance of limit diffusion when long-run player friendly

$\sigma_{-1} / \sigma_{+1} > 1$ a large draw of z “bad news”

$\sigma_{-1} / \sigma_{+1} < 1$ a large draw of z “good news”

equal variances $\sigma_{+1} = \sigma_{-1}$.

Proposition 7: *In the bad news case ($\sigma_{-1} / \sigma_{+1} > 1$) if $\lim_{\Delta \to 0} \tau(\Delta) / \Delta = \infty$ there is an efficient limit equilibrium.*

Binomial Arrays Converging to Diffusions

Proposition 9: *Suppose the period length is Δ, and that we have i.i.d. binomials $Z_i(\Delta)$ where the common outcomes are $x(\Delta) > y(\Delta)$ and the probability of $x(\Delta)$ under action i is $\alpha_i(\Delta)$, with $\lim_{\Delta\to 0}\alpha_i(\Delta) = \alpha_i, 0 < \alpha_i < 1$. If the sums $\sum_{j=1}^{[\![t/\Delta]\!]} Z_j$ converge to a diffusion with drift μ_i and volatilities σ_i^2 then $\sigma_1 = \sigma_2$.*

Proposition 10: *Suppose in addition*

(i) $\lim_{\Delta\to 0}\tau(\Delta)\exp(k(\Delta)^{2/7}) \to \infty$

Then all limit equilibria are trivial.

Trinomial Informational Limits

drifts μ_1, μ_{-1} and volatilities $\sigma_1^2, \sigma_{-1}^2$

construct a particular family of pairs of trinomials with aggregate converging to diffusion with these parameters

indexed by a free parameter γ not determined by the limit diffusions

for any $\gamma \geq 1$ set $\overline{\gamma} = \gamma \max(\sigma_1^2, \sigma_{-1}^2)$

three possible outcomes, $x = -h(\Delta), 0, h(\Delta)$, $h(\Delta) = \overline{\gamma}^{1/2}\Delta^{1/2}$

probability of outcome 0 $\alpha_i = (\overline{\gamma} - \sigma_i{}^2) / \overline{\gamma}$

probability of outcome $+h$

$$\beta_i(\Delta) = \frac{1-\alpha_i}{2} + \frac{\mu_i \Delta^{1/2}}{2\overline{\gamma}^{1/2}}.$$

Bad news case $\sigma^2_{-1} > \sigma^2_{+1}$ and Zero Means.

construct a sequence of games converging to a diffusion with common volatilies with an efficient limit equilibrium

conclusions based on hypothesis that the variances are equal in the limit do not apply to the limit of the equilibria along the sequence without additional information, such as the rate at which the variances become equal

Good News Case $\sigma^2_{-1} < \sigma^2_{+1}$ *and Zero Means*

$\tau(\Delta) = \Delta$

best limit equilibrium payoff i$\bar{u} - \frac{(\gamma - 1)\sigma^2_{-1}}{\sigma^2_1 - \sigma^2_{-1}} g$

$\tau(\Delta) = \Delta^{1/2}$

signals observed each period converge diffusions

suppose $\frac{\sigma_{+1} - \sigma_{-1}}{\sigma_{-1}} > g/(\overline{u} - \underline{u})$ so non trivial limit equilibrium for diffusion

γ large get examples with only trivial equilibrium when $\tau = \Delta$ and non trivial limit when $\tau(\Delta) = \Delta^{1/2}$

γ near 1, and σ_{-1} near σ_{+1} get examples with non trivial limit when $\tau = \Delta$ and trivial limit when $\tau(\Delta) = \Delta^{1/2}$