Econ 504 (2008) Microeconomic Theory II - Midterm I Solutions

1. Since messages do not directly affect payoffs, such a signaling game is also called a cheap talk game. In particular by symmetry of messages, for every sequential equilibrium (β, μ), the (β', μ') obtained by permuting the two messages is also a sequential equilibrium.

   First note that there is no separating equilibrium. Suppose for instance that t_A plays Left and t_B plays Right. Then the receiver would play Up after Left and Down after Right, in which case each type of sender would have a strict preference to deviate. By symmetry of messages, the other separating equilibrium also breaks down.

   Next note that the probabilities with which the receiver plays Up conditional on Left and Right must be the same. Otherwise t_A would strictly prefer to send the message that leads to the higher probability of Down and t_B would strictly prefer to send the other message, a contradiction to there not being a separating equilibrium. Let α denote the probability with which the receiver plays Up conditional on either message.

   Consider a pooling equilibrium on Left. Then $\mu(t_A|L) = 0.2$ implying $\alpha = 0$. For $\alpha = 0$ to be optimal conditional on Right off-equilibrium beliefs should satisfy $\mu(t_A|R) \leq 1/2$. Conversely any such assessment (and those obtained by permuting the messages) is a pooling sequential equilibrium. All such pooling equilibria trivially satisfy the intuitive criterion since no type's payoff from the pooling message strictly exceeds his maximum achievable payoff from the other message.

   Finally consider sequential equilibria where at least on type of sender mixes. Let a and b denote the probabilities with which t_A and t_B play Left respectively. Then:

$$\mu(t_A|L) = \frac{0.2a}{0.2a + 0.8b} \quad \text{and} \quad \mu(t_A|R) = \frac{0.2(1-a)}{0.2(1-a) + 0.8(1-b)} \tag{1}$$

   It can not be that $\mu(t_A|L), \mu(t_A|R) \geq 1/2$ because otherwise $a \geq 4b$ and $(1-a) \geq 4(1-b)$, a contradiction. Thus after some message, the receiver believes that he is facing t_B with probability strictly more than 1/2, implying that $\alpha = 0$. For $\alpha = 0$ to be optimal, we must have $\mu(t_A|L), \mu(t_A|R) \leq 1/2$, which by the above equations is equivalent to:

$$a \leq 4b \quad \text{and} \quad 1 - a \leq 4(1-b). \tag{2}$$

Conversely any $a \in [0,1]$, $b \in (0,1)$, satisfying Equation (2) and μ derived by the Bayes rule in Equation (1) constitute a sequential equilibrium.

2. The profit of firm i is concave in p_i. The first order condition $\frac{d}{dp_i}\pi_i(p_i, p_j) = 1 + p_j - 2p_i = 0$ implies that the best reply of firm i is $p_i^*(p_j) = \frac{1}{2}p_j + \frac{1}{2}$. We next prove that **if firm i knows that $p_j \geq \bar{p}_j$ for some fixed $\bar{p}_j$, then $p_i^*(\bar{p}_j)$ strictly dominates any $p_i < p_i^*(\bar{p}_j)$**. First note:

$$p_i^*(\bar{p}_j)(1 + 2\bar{p}_j - p_i^*(\bar{p}_j)) > p_i(1 + 2\bar{p}_j - p_i) \tag{3}$$

because $p_i^*(\bar{p}_j)$ is the unique best reply to $\bar{p}_j$ and $p_i \neq p_i^*(\bar{p}_j)$. Note also that:

$$p_i^*(\bar{p}_j)(2p_j - 2\bar{p}_j) \geq p_i(2p_j - 2\bar{p}_j) \tag{4}$$

because $p_i^*(\bar{p}_j) > p_i$ and $p_j \geq \bar{p}_j$. Summing Equations (3) and (4), we have:

$$p_i^*(\bar{p}_j)(1 + 2p_j - p_i^*(\bar{p}_j)) > p_i(1 + 2p_j - p_i).$$

That is , $\pi_i(p_i^*(\bar{p}_j), p_j) > \pi_i(p_i, p_j)$ as desired.

(a) Since $p_j \geq 0$, firm i is rational implies $p_i \geq p_i^*(0) = \frac{1}{2}$. So (iii) implies that: firm 1 knows that firm 2 knows that $p_1 \geq \frac{1}{2}$. Then, (ii) implies that firm 1 knows that $p_2 \geq p_2^*(\frac{1}{2}) = \frac{3}{4}$. Finally (i) implies that $p_1 \geq p_1^*(\frac{3}{4}) = \frac{7}{8}$.

(b) Define $x_0 = 0$ and $x_t = \frac{1}{2^t}\sum_{i=1}^{t} 2^{i-1}$ for $t = 1, 2, \ldots$. Note that if firm i knows $p_j \geq x_t$ for some $t \geq 0$, and if firm i is rational, then $p_i \geq p_i^*(x_t) = \frac{1}{2}x_t + \frac{1}{2} = x_{t+1}$. Therefore common knowledge of rationality implies $p_i \geq x_t$ for all t. Since $x_t \nearrow 1$, we must have $p_1 = p_2 = 1$. Since the outcome of IESDS is unique, it also corresponds to the unique Nash equilibrium of this game.

3. (a) Let v^α denote the continuation value of a player in an SPE prior to the realization of who makes an offer. By symmetry, we conjecture $v^\alpha = \frac{\alpha}{2}$.

Strategies: Each player i offers $(x_i, x_j) = (\alpha(1 - \frac{\delta}{2}), \alpha\frac{\delta}{2})$. Each player accepts if and only if he is offered at least $\alpha\frac{\delta}{2}$.

Verify the above strategies are SPE: Use the single deviation principle. At a history in which i makes an offer, the maximum payoff i can achieve by making an offer that will be immediately accepted is $\alpha(1 - \frac{\delta}{2})$. If i makes an offer that will be rejected by j, then he receives $\alpha\frac{\delta}{2}$ payoff in the continuation which is strictly less. So i does not have a profitable single deviation.

At a history where j is about to accept or reject, her continuation payoff from rejecting is $\alpha\frac{\delta}{2}$, so she also has no profitable single deviation.

(b) Let v denote the continuation value of a player in an SPE prior to the realization of who makes an offer. By symmetry, we conjecture $v = \frac{1}{3}$.

Derivation of Strategies: After a history with only two players and a remaining cake size α, players play as in the SPE in part (a).

First consider a history where i has offered (x_i, x_j, x_k) and j and k are about to accept or reject. Conditional on k rejecting, we conjectured j's payoff from rejection to be $\frac{\delta}{3}$. Conditional on k accepting, j's payoff from rejection is $(1-x_k)\frac{\delta}{2}$, since in this case next period j will start bargaining with i over the remaining cake of size $\alpha = 1 - x_k$.

Define the following subsets of $X = \{(x_j, x_k) \in [0,1]^2 : x_j + x_k \leq 1\}$:

$$\begin{aligned} X_j^+ &= \{(x_j, x_k) \in X : x_j \geq (1-x_k)\frac{\delta}{2}\} \\ X_j^- &= \{(x_j, x_k) \in X : x_j \leq \frac{\delta}{3}\} \end{aligned}$$

Symmetrically define X_k^+, X_k^-, let $Y = (X_j^+)^{\complement} \cap (X_j^-)^{\complement} \cap (X_k^+)^{\complement} \cap (X_k^-)^{\complement}$. Consider any strategy profile satisfying.:

- Both j and k accept if $(x_j, x_k) \in X_j^+ \cap X_k^+$.
- Exactly one of j or k accepts if $(x_j, x_k) \in Y$.
- j accepts and k rejects if $(x_j, x_k) \in [(X_j^-)^{\complement} \cap (X_k^+)^{\complement}] \setminus Y$.
- k accepts and j rejects if $(x_j, x_k) \in [(X_k^-)^{\complement} \cap (X_j^+)^{\complement}] \setminus Y$.
- Both j and k reject if $(x_j, x_k) \in X_j^- \cap X_k^-$.

Note that the intersection of $x_j = (1-x_k)\frac{\delta}{2}$ and $x_k = (1-x_j)\frac{\delta}{2}$ is $x_j = x_k = \frac{\delta}{2+\delta} > \frac{\delta}{3}$. Draw a picture to verify that the five regions considered above form a partition of X, which ensures that player j and k's replies to the offer (x_i, x_j, x_k) above are well-defined. Finally, each player i offers:

$$(x_i, x_j, x_k) = \left(\frac{2-\delta}{2+\delta}, \frac{\delta}{2+\delta}, \frac{\delta}{2+\delta}\right).$$

Verify the above strategies are SPE: After a history with only two players and a remaining cake size α we know from part (a) that the strategies constitute an SPE. For the rest of the argument consider histories with three players. We will again use the single deviation principle.

At a history in which i makes an offer, the maximum payoff i can achieve by making an offer that both of the other players will accept (i.e. s.t. $(x_j, x_k) \in X_j^+ \cap X_k^+$) is $\frac{2-\delta}{2+\delta} = 1 - 2\frac{\delta}{2+\delta}$. The supremum payoff i can achieve by making an offer that exactly one other player accepts (i.e. s.t. $(x_j, x_k) \in (X_j^+ \cap X_k^+)^{\complement} \cap (X_j^- \cap X_k^-)^{\complement}$) is $(1-\frac{\delta}{3})\frac{\delta}{2}$.[1] The payoff that i can achieve by making an offer that will be rejected by both other players is $\frac{\delta}{3}$. Since $\frac{2-\delta}{2+\delta} > (1-\frac{\delta}{3})\frac{\delta}{2}, \frac{\delta}{3}$; i has no profitable single deviation.

Next consider a history where i has offered (x_i, x_j, x_k) and j and k are about to accept or reject.

- If $(x_j, x_k) \in X_j^+ \cap X_k^+$, then j expects k to accept. Given this, it is optimal for j to accept if $x_j \geq (1-x_k)\frac{\delta}{2}$, i.e. if $x_j \in X_j^+$. The argument for k is symmetric.
- If $(x_j, x_k) \in Y$ and suppose w.l.o.g. that the strategies are such that j is accepts and k rejects. Since j expects k to reject, it is optimal for j to accept if $x_j \geq \frac{\delta}{3}$ that is if $(x_j, x_k) \in (X_j^-)^{\complement}$. Since k expects j to accept it is optimal for k to reject if $x_k < (1-x_j)\frac{\delta}{2}$, i.e. if $(x_j, x_k) \in (X_k^+)^{\complement}$.
- If $(x_j, x_k) \in [(X_j^-)^{\complement} \cap (X_k^+)^{\complement}] \setminus Y$, the argument is the same as the one above.
- If $(x_j, x_k) \in [(X_k^-)^{\complement} \cap (X_j^+)^{\complement}] \setminus Y$, the argument is symmetric to the one above.
- If $(x_j, x_k) \in X_j^- \cap X_k^-$, then j expects k to reject. Given this, it is optimal for j to reject if $x_j \leq \frac{\delta}{3}$ i.e. if $(x_j, x_k) \in X_j^-$. The argument for k is symmetric.

[1] This is achieved by offering $x_k = \frac{\delta}{3} + \epsilon$ and $x_j < \frac{\delta}{3}$, after which k exits the game with $x_k = \frac{\delta}{3} + \epsilon$, and the next period i and j start bargaining over the remaining cake of size $\alpha = 1 - \frac{\delta}{3} - \epsilon$.