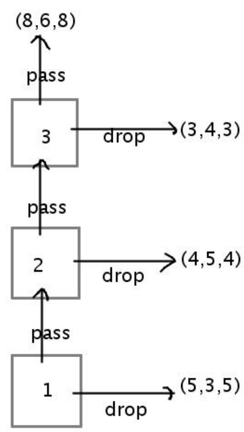
ECON 201B - Game Theory

Suggested Answers - Midterm 1

February 2, 2006

1 First Problem

Consider the extensive form game given below



a) Find all subgame perfect equilibria.

To obtain SGPE we need to work backwards. Player 3 will prefer to pass (P_3) (obtaining 8 rather than 3). Knowing this, player 2 will also prefer to pass (P_2) (obtaining 6 rather than 5). Finally player 1 will prefer to pass as well (P_1) (obtaining 8 rather than 5). Hence the unique SGPE is (P_1, P_2, P_3)

b) Find the normal form.

Player 1 picks matrices, player 2 rows and player 3 columns. Hence the normal form is (bold-face numbers denote best responses)

$\mathbf{D}_1$

	D_3	P_3
D_2	$\mathbf{5}, \mathbf{3}, \mathbf{5}$	$\mathbf{5}, \mathbf{3}, \mathbf{5}$
P_2	$\mathbf{5}, \mathbf{3}, \mathbf{5}$	$5, \mathbf{3}, \mathbf{5}$

$\mathbf{P}_1$

	D_3	P_3
D_2	$4, \mathbf{5}, \mathbf{4}$	$4, 5, \mathbf{4}$
P_2	$3, 4, 3$	$\mathbf{8}, \mathbf{6}, \mathbf{8}$

c) Apply iterated strict dominance.

There is no strategy strictly dominated for any player (there is no strategy strictly preferred by a player regardless of what other players do).

However, recall P_3 is weakly preferred by player 3.

d) Find all Nash equilibria, including mixed equilibria.

From the normal form game, there are four pure strategy NE.

$\{(D_1, D_2, D_3); (D_1, P_2, D_3); (D_1, D_2, P_3); (P_1, P_2, P_3)\}$

To obtain mixed strategy NE, we can use the hard way (obtaining best responses in mixed strategies and checking for consistency) or use an easier way (based on a logical iteration to eliminate possibilities).

Hard way (Consider $\sigma_1 = \Pr(D_1)$, $\sigma_2 = \Pr(D_2)$, $\sigma_3 = \Pr(D_3)$)

1) Best response for Player 1:

Player 1 will play D_1 ($\sigma_1 = 1$) if:

$$5\sigma_2\sigma_3 + 5\sigma_2(1-\sigma_3) + 5(1-\sigma_2)\sigma_3 + 5(1-\sigma_2)(1-\sigma_3) >$$
$$4\sigma_2\sigma_3 + 4\sigma_2(1-\sigma_3) + 3(1-\sigma_2)\sigma_3 + 8(1-\sigma_2)(1-\sigma_3)$$

Hence:

$\sigma_1 = 1$ if $5 > (8 - 5\sigma_3) - \sigma_2(4 - 5\sigma_3)$

$\sigma_1 = 0$ if $5 < (8 - 5\sigma_3) - \sigma_2(4 - 5\sigma_3)$

$\sigma_1 \in [0, 1]$ if $5 = (8 - 5\sigma_3) - \sigma_2(4 - 5\sigma_3)$

2) Best response for Player 2:

Player 2 will play D_2 ($\sigma_2 = 1$) if:

$5(1 - \sigma_1) > (1 - \sigma_1)[4\sigma_3 + 6(1 - \sigma_3)]$

Hence:

$\sigma_2 = 1$ if $\sigma_3 > \frac{1}{2}$ and $\sigma_1 < 1$

$\sigma_2 = 0$ if $\sigma_3 < \frac{1}{2}$ and $\sigma_1 < 1$

$\sigma_2 \in [0, 1]$ if $\sigma_3 = \frac{1}{2}$ and/or $\sigma_1 = 1$

3) Best response for Player 3:

Player 3 will play D_3 ($\sigma_3 = 1$) if:

$3(1 - \sigma_1)(1 - \sigma_2) > 8(1 - \sigma_1)(1 - \sigma_2)$

Hence:

$\sigma_3 = 0$ if $\sigma_1 \in [0, 1)$ and $\sigma_2 \in [0, 1)$

$\sigma_3 \in [0, 1]$ if $\sigma_1 = 1$ and/or $\sigma_2 = 1$

Obtaining mixed Nash Equilibria

Consider the case $\sigma_1 = 1$. The b.r. would be $\sigma_2 \in [0,1]$ for player 2 and $\sigma_3 \in [0,1]$ for player 3.

Now we have to double check if $\sigma_1 = 1$ is still optimal for player 1. This is the case only if $5 > (8 - 5\sigma_3) - \sigma_2(4 - 5\sigma_3)$

For example, if $\sigma_2 = 0$, the condition becomes $5 > 8 - 5\sigma_3$ or $\sigma_3 > \frac{3}{5}$. This equilibria includes the pure strategy NE (D_1, P_2, D_3)

For example, if $\sigma_2 = 1$, the condition becomes $5 > 4$, what happens for any $\sigma_3 \in [0,1]$. This equilibria includes the pure strategy NE (D_1, D_2, P_3) and (D_1, D_2, D_3)

In general, the mixed Nash Equilibria when player 1 drops for sure can be expressed as $\boxed{\sigma_1 = 1,\ \sigma_2 > \frac{3-5\sigma_3}{4-5\sigma_3} \text{ and } \sigma_3 \in [0,1]}$

Consider the case $\sigma_1 \in [0,1]$. This happens only when $\sigma_2 = \frac{3-5\sigma_3}{4-5\sigma_3}$.

It's easy to observe that σ_2 cannot be 1 for any $\sigma_3 \in [0,1]$

Other case is $\sigma_2 = 0$ which is possible only when $\sigma_3 = \frac{3}{5}$. This is not an equilibrium since the b.r. for player 2 when $\sigma_3 = \frac{3}{5}$ and $\sigma_1 \in [0,1)$ is $\sigma_2 = 1$. The only equilibrium in this case is $\sigma_1 = 1$, $\sigma_2 = 0$ and $\sigma_3 = \frac{3}{5}$

Finally, $\sigma_2 \in [0,1]$ is player 2's b.r. for $\sigma_1 \in [0,1)$ if $\sigma_3 = \frac{1}{2}$. This implies that $\sigma_2 = \frac{1}{3}$. But in this case the optimal play for 3 is $\sigma_3 = 0$. Hence the only mixed equilibrium in this case is $\sigma_1 = 1$, $\sigma_2 = \frac{1}{3}$ and $\sigma_3 = \frac{1}{2}$

Hence in general, player 1 cannot randomize in the mixed Nash equilibria and $\boxed{\sigma_1 = 1,\ \sigma_2 = \frac{3-5\sigma_3}{4-5\sigma_3} \text{ and } \sigma_3 \in [0, \frac{3}{5}]}$

Consider the case $\sigma_1 = 0$. This happens only when $\sigma_2 < \frac{3-5\sigma_3}{4-5\sigma_3}$.

In this case σ_2 cannot be 1, since then $5 > 4$ and the best response for player 1 should be $\sigma_1 = 1$ and not 0.

The case in which $\sigma_2 \in (0,1)$ cannot be an equilibrium since $\sigma_3 = \frac{1}{2}$ and then from the condition above, $\sigma_2 < \frac{1}{3}$, which paired with $\sigma_1 = 0$ implies that σ_3 should be 0 and not a half.

In the case $\sigma_2 = 0$, from the b.r. condition for player 1 $\sigma_3 < \frac{3}{5}$. In fact, since $\sigma_1 = \sigma_2 = 0$, it's necessary from the b.r for player 3 that $\sigma_3 = 0$.

Hence this equilibrium is $\boxed{\sigma_1 = \sigma_2 = \sigma_3 = 0}$ which is the one in pure strategies (P_1, P_2, P_3) and in fact the SGPE found before.

Easier way

In the case Player 1 decides to drop, both players 2 and 3 are indifferent between strategies (they always get the same, 3 and 5 respectively). But naturally there is a restriction in the way they can randomize since Player 1 should have decided optimally to drop. Hence we need to ask which is the combination of probabilities σ_2 and σ_3 that give player 1 less than the 5 he can get by dropping. (i.e. $5 \geq 4\sigma_2 + (1-\sigma_2)(3\sigma_3 + 8(1-\sigma_3))$ which determines the condition $\sigma_2 \geq \frac{3-5\sigma_3}{4-5\sigma_3}$).

But, if player 1 decides to randomize, then player 2 will eventually play. In this case he will drop only if the probability player 3 drops is high enough (i.e., if $\sigma_3 > \frac{1}{2}$, the expected payoff from passing is smaller than the 5 player 2 can obtain from dropping). But this is not an equilibrium because if player 3 decides to drop, player 1 would decide to drop in the first place.

Player 3 will randomize only in the case either player 1 or player 2 drops and not otherwise. If both players 1 and 2 randomize with some probability, player 3 would decide to pass for sure, and considering this it's not optimal for player 2 to randomize. Hence the only possible equilibrium when 1 passes is such that everybody passes.

e) Find a self-confirming equilibrium with an equilibrium path that cannot be obtained as a public randomization over Nash equilibria.

A self confirming equilibrium that has an equilibrium path not attainable by a public randomization over NE is $(P_1, \frac{1}{2}D_2 + \frac{1}{2}P_2, P_3)$. This a heterogenous self confirming equilibrium that has a path (namely player 2 dropping when player 1 passes) that is not a Nash equilibrium (since when 1 passes, the only equilibrium is that 2 and 3 also pass, as was shown above).

The key here is that some player 2's have correct beliefs about the decision of player 3 and passes while other player 2's fear that 3 drops and so chose not to give 3 the chance to move, dropping and preventing them from learning that their beliefs are mistaken.

Nevertheless, if the proportion of these two "types" of player 2 is 50-50, player 1 will still prefer to play pass, having a expected utility of 6, greater than the 5 from dropping.

2 Second Problem

Consider the following normal form game (bold-face numbers denote best responses)

	L	R
U	$0,0$	$\mathbf{2,1}$
D	$\mathbf{1,2}$	$0,0$

a) Find all Nash equilibria

There are two pure strategy NE. It's possible to find a mixed strategy NE as well.

Consider $\sigma_1 = \Pr(U)$ and $\sigma_2 = \Pr(L)$

Player 1 will randomize if $2(1-\sigma_2) = \sigma_2 \Longrightarrow \sigma_2 = \frac{2}{3}$. By symmetry $\sigma_1 = \frac{2}{3}$

Hence, the set of NE is $\{(U,R);(D,L);(\frac{2}{3}U+\frac{1}{3}D, \frac{2}{3}L+\frac{1}{3}R)\}$

The corresponding payoffs in these NE are $\{(2,1),(1,2),(\frac{2}{3},\frac{2}{3})\}$

b) Which Nash equilibria are trembling hand perfect?

All of them since there are two strict NE and a completely mixed strategy NE.

c) Is there a correlated equilibrium that gives strictly less utility than any public randomization over Nash equilibrium.

First it's important to say that the minimum payoff achievable with any public randomization over Nash equilibrium is $\frac{2}{3}$, (the payoff obtained from the worst equilibrium, which is the mixed strategy NE). Any randomization between the three equilibria would deliver a payoff no less than $\frac{2}{3}$ and no greater than 2.

Let's start considering a correlated device that also represents the mixed strategy NE (which delivers the worst payoffs).

	L	R
U	4/9	2/9
D	2/9	1/9

Since the game as well as the proposed correlated equilibria are both symmetric we only need to check that the incentive constraints are satisfied for one player. Take player 1. If recommended by the mediator to play U, and player 1 obeys the mediator, her payoff will be (using the posterior probability updated from the mediator's recommendation) $0.\frac{2}{3} + 2.\frac{1}{3} = \frac{2}{3}$, which is exactly equal to the payoff from disobeying by playing D, $1.\frac{2}{3} + 0.\frac{1}{3} = \frac{2}{3}$. Similarly, if the mediator recommended D, either in the case player 1 obeys or disobeys, her payoff will be $\frac{2}{3}$.

In order to get less utility we would need to put more weight on the cases (U, L) or (D, R) that have zero payoffs. But to maintain the correlated equilibrium, the probability 2 is recommended to play L when 1 is recommended to play U ($\Pr(L|U)$) cannot be greater than $\frac{2}{3}$, otherwise player 1 would prefer to deviate from the recommendation U. In the same vein, $\Pr(R|D)$ cannot be greater than $\frac{1}{3}$.

Hence there is a restriction in the weight we can assign to zero payoffs. The limit is given exactly by the randomization generated by the mixed strategy NE, which gives utility $(\frac{2}{3}, \frac{2}{3})$. Hence there is no correlated equilibrium that gives STRICTLY LESS utility than ANY public randomization over Nash equilibrium.