ECON 201B - Game Theory

Suggested Answers - Final Exam

March 24, 2006

1 Hunter-Gatherer

Two players must decide whether to be hunters or gathers. If both are hunters, both receive 0*; if both are gatherers both receive* 1*. If one is a hunter and one a gatherer, the hunter receives* 3 *and the gatherer* 2*.*

a) *Find the normal form of this game.*

	H	G
H	$0, 0$	$\mathbf{3}, \mathbf{2}$
G	$\mathbf{2}, \mathbf{3}$	$1, 1,$

b) *Find the Nash equilibrium of this game.*

Best responses in pure strategies are denoted in bold in the matrix above. There are two NE in pure strategies (H, G) and (G, H). There is also a mixed NE in which both players randomizes 50-50 (i.e. $(\frac{1}{2}G + \frac{1}{2}H, \frac{1}{2}G + \frac{1}{2}H)$). **NE Payoffs to player 1 in each case are respectively 3, 2 and 1.5**.

c) *Are there any dominated strategies?*

There is **NO** strategy strictly dominated (i.e. no strategy strictly preferred by a player regardless of what the other player plays).

d) *Find the pure and mixed Stackelberg equ. in which player 1 moves first.*

The highest NE payoff is also the highest possible payoff attainable by player 1 (i.e. 3). Hence both the **pure and mixed Stackelberg equilibrium deliver 3** as well (when player 1 commits to play H and player 2 reacts by playing G).

e) *Find the minmax for both players.*

The definition of minmax (for player 1) is $\mathbf{m^1} = \min_{\alpha^2} \left[\max_{a^1} u^1(a^1, \alpha^2)\right]$. In this game

α^2	$a^1 \in BR^1(\alpha^2)$	Payoffs
$Pr(H) < \frac{1}{2}$	G	$(1.5, 2]$
$Pr(H) = \frac{1}{2}$	G, H, mix	1.5
$Pr(H) > \frac{1}{2}$	H	$(1.5, 3]$

being $\mathbf{m^1}= \mathbf{1.5}$ the worst possible payoff for player 1 when player 2 precommits to randomizes 50-50 (i.e., the mixed NE). Since the game is symmetric, the same is true for player 2.

Now suppose that the game is infinitely repeated

f) *Player 1 is a long-run player with discount factor δ; player 2 is a short-run player with discount factor 0. Find the set of perfect public equilibrium payoffs to the long-run player as a function of her discount factor.*

The minmax to player 1 delivers the same payoff of 1.5 than the worst possible NE (the mixed one), then, the worst dynamic payoff $\underline{\boldsymbol{v}}^{\mathbf{1}}$ **is** $\mathbf{1.5}$

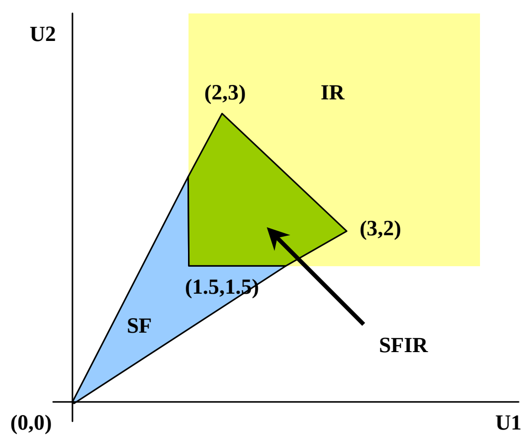
Both pure and mixed Stackelberg equilibria give the same payoff of 3, then, the best dynamic payoff $\overline{\boldsymbol{v}}^{\mathbf{1}}$ **is** $\mathbf{3}$.

Hence, the set of dynamic equilibria is composed by all the payoffs between 1.5 and 3. Since these two extreme cases correspond to static NE, this is true for all discount factors.

g) *Find strategies that support the best equilibrium from part f.*

Play the static NE (H, G) every period, getting 3 always

h) *Player 1 and 2 are both long-run players with common discount factor δ. When close to 1, describe the set of perfect equilibrium payoffs to both players.*



i) *Find a discount factor and strategies for part h such that both players receive an equilibrium payoff of 2.5.*

A payoff of 2.5 can be achieved for all discount factors just by public randomizing 50-50 between the static NE given by (H, G) and (G, H)

2 Greenspan

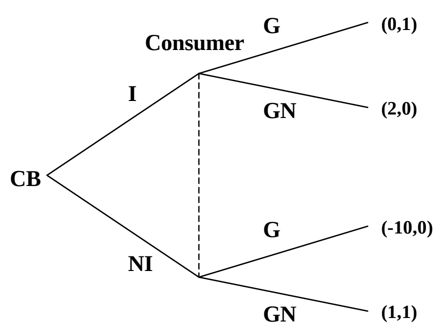
A long-lived central bank faces a short-run representative consumer. The bank must decide whether or not to inflate; the consumer must decide whether or not to expect inflation. If the consumer guesses correctly, she gets 1; incorrectly she gets 0. Central bank payoffs are

	Guess inflate (G)	Guess Not (GN)
Inflate (I)	0	2
Not inflate (NI)	-10	1

As a result of whether or not the central bank chose to inflate, economic activity is determined: there are two possibilities hyperinflation or price stability. If the bank chose to inflate the probability of hyperinflation is 1; if the bank chose not to inflate, the probability of hyperinflation is 10%. In all that follows, equilibrium *means perfect public equilibrium of the infinitely repeated game with public randomization.*

a) *Find the extensive and normal forms of the stage-game.*

The extensive form of the stage-game is,



and the stage game in normal form is,

	G	GN
I	$0, 1$	$2, 0$
NI	$-10, 0$	$1, 1$

b) *For the long-run player, find the minmax, the static Nash, mixed precommitment and pure precommitment payoffs.*

First, inflating (I) is a dominant strategy to the Central Bank. Hence, it's straightforward to see that (I, G) is the unique static NE, with a payoff of 0 to player 1. This is also the minmax.

To obtain the Stackelberg equilibrium we must first analyze player 2's best response to each possible player 1's movement.

α^1	$a^2 \in BR^2(\alpha^1)$	Payoffs
$Pr(I) = 0$	GN	1
$0 < Pr(I) < \frac{1}{2}$	GN	$(1, 1.5)$
$Pr(I) = \frac{1}{2}$	G, GN, mix	$[-5, 1.5]$
$\frac{1}{2} < Pr(I) < 1$	G	$(-5, 0)$
$Pr(I) = 1$	G	0

Recall the definition of a mixed Stackelberg equilibrium is

$$\mathbf{ms^1} = \max_{(\alpha^1,\alpha^2)|\alpha^2 \in BR^2(\alpha^1)} \sum_{a^1} u^1(\alpha^1, \alpha^2)\alpha^1(a^1)$$

Hence we need to take the maximum value among the possible payoffs that can be obtained for each α^1. In this case this is 1.5 (when player 1 commits to randomize 50-50 and player 2 plays GN).

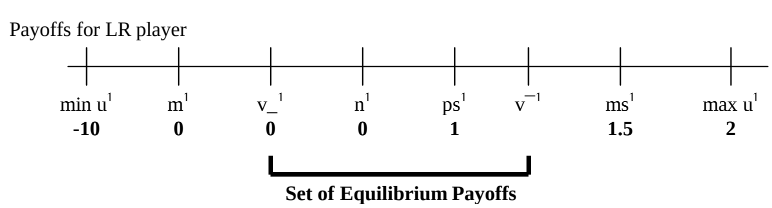
A pure Stackelberg equilibrium is $\mathbf{ps^1} = \max_{(a^1,\alpha^2)|\alpha^2 \in BR^2(a^1)} u^1(a^1, \alpha^2)$

This is the same definition than mixed Stackelberg but restricting attention to player 1's pure commitments. Therefore we need to take the maximum just comparing the first and last rows, which is 1 in this case.

c) *Find the worst equilibrium for the long-run player, and describe in general terms the set of equilibrium payoffs for the long-run player.*

As seen, both static NE and minmax payoffs are 0. Hence $\underline{v}^1 = 0$. In general the set of equilibrium payoffs is a line segment from 0 to $\overline{v}^1$

Summarizing



First assume that the consumer can observe whether or not the central bank inflates.

d) *Find the best equilibrium for the central bank as a function of the discount factor.*

The best equilibrium payoff $\overline{v}^1$ is given by

$$\boldsymbol{\overline{v}^1} = \max_{(\alpha^1,\alpha^2)|\alpha^2 \in BR^2(\alpha^1)} \left[\min_{a^1|\alpha^1(a^1)>0} u^1(a^1,\alpha^2) \right]$$

In this case, (for a δ large enough).

α^1	$a^2 \in BR^2(\alpha^1)$	Worst in Support
$Pr(I) = 0$	GN	1
$0 < Pr(I) < \frac{1}{2}$	GN	1
$Pr(I) = \frac{1}{2}$	G, GN, mix	$[-10, 1]$
$\frac{1}{2} < Pr(I) < 1$	G	-10
$Pr(I) = 1$	G	0

Maximizing over a^2, $\boldsymbol{\overline{v}^1 = 1}$

For the best equilibrium $\overline{v}^1$ to be 1 under Nash threats, we need $1 \geq (1-\delta)2 + \delta 0$, or which is the same $\boldsymbol{\delta} \geq \frac{1}{2}$

Now assume that the consumer cannot observe whether or not the central bank inflates but can observe whether or not there is hyperinflation.

e) *Find the best equilibrium for the central bank as a function of the discount factor.*

In this case, the consumer can only observe price conditions as a signal about the Central Bank action. Conditional probabilities are $\Pr(H|NI) = 0.1$ and $\Pr(H|I) = 1$. Continuation values will be now functions of signals and not actions. For example, $w(H)$ will be the continuation utility after an hyperinflation and $w(NH)$ after a price stability.

Consider the case in which player 2 "Guess Not" (GN). This is relevant since it represents how player 1 would like player 2 to play. But in order GN to be a best response by 2, it would be the case that player 1 plays I.

Using general notation, the set of equations to solve will be,

If player 1 plays *NI,*

$\overline{v}^1 = (1-\delta)u^1_{(NI,GN)} + \delta[\Pr(H|NI)w(H) + \Pr(NH|NI)w(NH)]$

If player 1 plays I,
$\overline{v}^1 \geq (1-\delta)u^1_{(I,GN)} + \delta[\Pr(H|I)w(H) + \Pr(NH|I)w(NH)]$
and finally,
$\underline{v}^1 \leq w(H), w(NH) \leq \overline{v}^1$

Naturally $w(NH) > w(H)$ and then $w(NH) = \overline{v}^1$. Since increasing $w(H)$ also increases $\overline{v}^1$, the second equation should hold with equality. Considering this and replacing conditional probabilities, we can rewrite the problem as,

$$\overline{v}^1 = (1-\delta) + \delta[(0.1)w(H) + (0.9)\overline{v}^1] \tag{1}$$

$$\overline{v}^1 = (1-\delta)2 + \delta w(H) \tag{2}$$

$$\underline{v}^1 \leq w(H) \tag{3}$$

Solving (1) and (2) (2 equations and 2 unknowns), we can get $\boldsymbol{\overline{v}^1}=\frac{8}{9}$ and $\mathbf{w(H)} = \mathbf{2}-\frac{10}{9\delta}$

Now, checking the inequality (3) we can get the δ such that $\overline{v}^1$ can be sustained. Hence $w(H) = 2 - \frac{10}{9\delta} \geq 0$, or which is the same $\boldsymbol{\delta} \geq \frac{5}{9}$

3 Mechanism Design

A risk averse consumer with utility has equal probability of endowment 1 or 20. A risk neutral insurance company offers a contract based on the statement of the consumer about her endowment. A consumer with a high endowment may misrepresent and pretend to have a low endowment. A consumer with a low endowment may not misrepresent. After the endowment is realized, the insurance company discovers the type (endowment) of the consumer with probability π, and if the type is observed may impose a penalty on the consumer. However, regardless of the state and the contract, the consumer may always "run away" and consume 1. What is the optimal contract?

The risk neutral insurance company offers a contract that gives money to low endowment persons (say x) and takes money away from high endowment persons (say y) trying to maximize her expected profits. The consumer would eventually be willing to participate in this insurance scheme because he's risk averse and wants to smooth consumption.

The problem differs from the typical insurance problem in being based on consumer's announcements rather than on observed endowment realizations.

Naturally a high endowment person would have incentives to lie in order the company giving money instead of taking it.

Hence the problem of the insurance company can be expressed as maximizing the difference between y and x subject to a Participation Constraint (PC)

$$\log(20 - y) + \log(1 + x) \geq \log(20) \tag{4}$$

and an Incentive Compatibility Constraint (IC)

$$\log(20 - y) \geq (1 - \pi) \log(20 + x) + \pi \log(20 - p) \tag{5}$$

Equation (4) means the expected utility with insurance ($\frac{1}{2} \log(20 - y) + \frac{1}{2} \log(1 + x)$) should be greater or equal the expected utility without insurance ($\frac{1}{2} \log(20) + \frac{1}{2} \log(1)$)

Equation (5) implies that a consumer that found his endowment is high after the contract has been signed does not have incentives to lie on his report. In this sense, the utility from saying the truth ($\log(20-y)$) is greater or equal that the expected utility from lying ($(1 - \pi) \log(20 + x) + \pi \log(20 - p)$), where π is the probability of being caught lying and p the penalty.

Regardless of the contract or the state the consumer may always "run away" and consume 1, even when he's supposed to pay the penalty. This imposes a restriction on the feasible p. Specifically, $\log(20-p) \geq \log(1) = 0$. The insurance company wants to penalize the consumer in a way he prefers to pay rather than just "run away".

Hence, $p = 19$. Plugging it into (5), the problem can be expressed as

$$\max \left[\frac{y - x}{2} \right] \tag{6}$$

subject to

$$\log(20 - y) + \log(1 + x) \geq \log(20) \tag{PC}$$

$$\log(20 - y) \geq (1 - \pi) \log(20 + x) \tag{IC}$$

Forming the Lagrangian

$$\mathcal{L} = \max \left\{ \begin{array}{c} \frac{y-x}{2} + \lambda_1 \left[\log(20 - y) + \log(1 + x) - \log(20)\right] \\ +\lambda_2 \left[\log(20 - y) - (1 - \pi) \log(20 + x)\right] \end{array} \right\}$$

FOC:

$\{y\}: \frac{1}{2} \leq \frac{\lambda_1+\lambda_2}{20-y}$ (= if $y > 0$)

$\{x\}: \frac{\lambda_1}{1+x} \leq \frac{1}{2} + \frac{\lambda_2(1-\pi)}{20+x}$ (= if $x > 0$)

$\{\lambda_1\}: PC$ (= if $\lambda_1 > 0$)

$\{\lambda_2\}: IC$ (= if $\lambda_2 > 0$)

We will focus on the case where both y and x are positive (in fact $y > x > 0$). Naturally this is the interesting case with insurance. It's easy to check λ_1 cannot be zero (PC always bind) while λ_2 can be zero (IC not necessarily binds). Then we have two possible cases:

- $\lambda_2 = 0$ (IC does not bind)

$$\lambda_1 = 10 - \frac{y}{2}$$

$$\frac{\lambda_1}{1+x} = \frac{1}{2}$$

$$\log(20-y) + \log(1+x) = \log(20)$$

$$\log(20-y) \geq (1-\pi)\log(20+x)$$

Solving this system of equations, $\mathbf{x = 3.47}$, $\mathbf{y = 15.53}$ and expected profits are 6.03. This is the optimal contract whenever $\boldsymbol{\pi} \geq \mathbf{0.525}$ (from last inequality).

- $\lambda_2 > 0$ (IC does bind)

$$\lambda_1 + \lambda_2 = 10 - \frac{y}{2}$$

$$\frac{\lambda_1}{1+x} = \frac{1}{2} + \frac{\lambda_2(1-\pi)}{20+x}$$

$$\log(20-y) + \log(1+x) = \log(20)$$

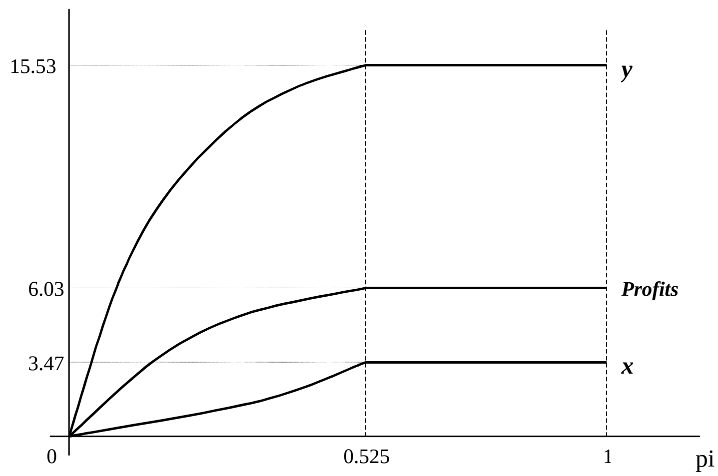
$$\log(20-y) = (1-\pi)\log(20+x)$$

From this system of equations, **x solves the equation**

$$(1-\pi)\log(20+x) + \log(1+x) = \log(20)$$

and $\mathbf{y = 20 - (20+x)^{1-\pi}}$. This is the optimal contract for "catching lies" probabilities $\boldsymbol{\pi} < \mathbf{0.525}$

It's possible to graph the optimal contract on y and x and the profits obtained for each possible probability π of detecting false reports



This figure is consistent with the results we know from the two extremes cases. When $\pi = 0$ (i.e. there is no way to know if the consumer lie or not), the insurance scheme is not implementable ($y = x = 0$). When $\pi = 1$ (i.e. the insurance company can observe the real endowment), we have the standard situation where the company maximizes the difference $y - x$ while charging up to the point in which the consumer's expected utility with insurance is equal to the expected utility without insurance.

4 Risk Aversion

a) *Starting from the expression* $u(x-p) = Eu(x+\sigma y)$ *with E(y)=0,* $E(y^2) = 1$*, derive the standard expression for the risk premium* p

Using a Taylor Series Expansion around x

$$u(x) - u'(x)p = E[u(x) + u'(x)\sigma y + \frac{1}{2}u''(x)\sigma^2 y^2]$$

Distributing the expectation

$$u(x) - u'(x)p = u(x) + u'(x)\sigma E(y) + \frac{1}{2}u''(x)\sigma^2 E(y^2)$$

Considering *E(y)=0* and $E(y^2) = 1$ and solving for p

$$p = -\frac{u''(x)}{u'(x)}\frac{\sigma^2}{2} \tag{7}$$

This is the absolute risk premium

b) *Suppose an individual is indifferent between getting nothing and a win \$105, lose \$100 equal probability gamble. For an individual with CES preferences, find the coefficient of relative risk aversion as a function of wealth, using the approximation of part a.*

The CES function is

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

where γ is a parameter that represents the constant relative risk aversion that characterizes CES functions (also known as CRRA functions)

Hence, $u'(x) = x^{-\gamma}$ and $u''(x) = -\gamma x^{-\gamma-1}$. Plugging them into (7),

$$p = \frac{\gamma}{x}\frac{\sigma^2}{2}$$

We can express the coefficient of relative risk aversion (γ) as a function of wealth (x) (in general notation) just by rearranging the previous expression,

$$\gamma = \frac{2p}{\sigma^2}x \tag{8}$$

In this particular case it's necessary to get p and σ^2 from the data. Using $u(E(x) - p) = u(0) = \frac{1}{2}u(105) + \frac{1}{2}u(-100)$, we can get

$$p = E(x) = \frac{105 - 100}{2}$$

or which is the same $p = 2.5$

By definition, the variance of the lottery is

$$\sigma^2 = E\left[(x-p)^2\right] = \frac{1}{2}\left[(105-2.5)^2 + (-100-2.5)^2\right]$$

which implies $\sigma^2 = (102.5)^2$

Plugging these results into (8) we get

$$\gamma = \frac{5}{(102.5)^2}x \tag{9}$$

c) *If wealth is \$350,000, what is the coefficient of relative risk aversion?*

From (9), if $x = 350000$, then $\gamma = 166.57$

d) *If preferences are logarithmic what is wealth? For what measure of wealth does the answer in part c make sense?*

A logarithmic utility $u(x) = \log(x)$ is a CES function where $\gamma = 1$. From (9), if $\gamma = 1$, then $x = 2101.25$