A Theory of the Dynamics of Factor Shares*

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Abstract

We show that, at medium-run frequencies, factor shares have a cyclical pattern correlated to that of productivity, employment, output and wages. We offer an interpretation of such cycles within the context of an equilibrium model of endogenous growth. Apart from standard features, common to most endogenous growth models, ours replicates the observed relations between factor shares and other macro variables. The dynamics of income distribution is driven by the interplay between opportunities for technological change and entrepreneurs seeking profit maximization. The interaction between factor prices and opportunities for labor-saving innovations "causes" both persistent growth and the factor shares oscillations.

JEL classification: E25, O30, O40

Keywords: Factor shares; Competitive innovation; Labor saving technical change; Growth cycles

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1 Introduction

We use data from the US and other advanced economies to show that, at medium-run frequencies, factor shares have a persistent cyclical pattern that displays clear correlations with movements in productivity, employment, output and real wages. Next, we propose an interpretation of such cycles within the context of a stylized general equilibrium model of endogenous growth.

Apart from the standard features of growth models, our theoretical framework mimics well the observed relations between factor shares and other macro variables. In our theory, the dynamics of income distribution is driven by the interplay between opportunities for technological change and the income maximization efforts of capital and labor. The interaction between factor prices and the opportunities for labor-saving innovation causes *both* persistent growth and factor shares oscillations. To put it plainly: when economic growth is generated by labor saving innovations, it brings about oscillations in the factor shares of national income similar to those observed in the data of advanced economies.

The causal mechanism driving the aggregate distribution of national income between capital and labor has been an enduring source of controversy among economists, and still is.[1] For many decades - since Nicholas Kaldor convinced the profession that constant factor shares were a "stylized fact" - macroeconomists have assumed that production functions are Cobb-Douglass, labor receives about 2/3 of national income and productivity grows forever at a constant rate. With remarkably few exceptions this state of affairs continued until the work of Piketty and coauthors (Piketty, 2014; Piketty and Zucman, 2014) broke the spell.[2] The ensuing debate has taken a variety of very interesting turns, and the decreasing trend hypothesis has been widely questioned.[3] Nevertheless, two things are apparent, which are the object of our investigation: (i) factor shares oscillate quite regularly at medium-run frequencies and, (ii) a good model for such movements is not yet available.

Our analysis of the data show that such oscillations exhibit the following regularities

- The growth rates of the labor share and of labor productivity are negatively correlated.
- The growth rates of the labor share and of real wages are positively correlated.

[1]A few relatively recent references include Hicks (1932), Kaldor (1957), Sraffa (1960), NBER (1964), Bronfenbrenner (1971), Foley and Michl (1999), Hansen and Prescott (2002), Piketty (2014), Sandmo (2015), Acemoglu and Restrepo (2018), Grossman and Oberfield (2021).

[2]Solow (1958) is among the earliest to raise a skeptical view on the constancy of factor shares, Boldrin and Horvath (1995) and Ríos-Rull and Santaeulalia-LIopis (2010) are among the more recent ones.

[3]See, e.g., Elsby et al. (2013), Karabarbounis and Neiman (2014), Rognlie (2015), Gutierrez and Piton (2020), Oberfield and Raval (2021), and Koh et al. (2021) on how measurement issues affect our understanding of such trends.

- The growth rate of the labor share is positively correlated with those of employment and of hours worked.
- The growth rates of the labor share and value added are weakly negatively correlated.

The stylized growth model we propose display regular medium-run oscillation satisfying all these regularities. To do this, we show in section 3 that the model with exogenous labor supply captures the first two facts and then, in section 4, derive the remaining two correlations by introducing endogenous labor supply.

Building on the notion of competitive innovation (Boldrin and Levine, 2001, 2008), our theory formalizes a mechanism through which the equilibrium dynamics of labor share, wage, and labor productivity become consistent with the observed patterns. Specifically, we construct a vintage capital model in which production of the final consumption good requires two complementary inputs, capital and labor. Technical progress, which is labor saving, is embodied in capital goods: machines of a more recent vintage require less labor to produce one unit of the final good. Apart from consumption, each vintage of capital can either reproduce itself or create capital of the next vintage. Innovation is costly, hence profit maximization determines whether investing should be in old or new machines[4].

The model economy admits a unique equilibrium path along which it settles into recurring cycles, each consisting of an *adoption phase* and an *innovation phase*. During the first, machines of two consecutive vintages are simultaneously employed in producing the final good, and labor is reallocated from the less to the more advanced vintage. This labor reallocation process increases both output and average labor productivity. At the same time, the real wage remains stagnant as the productivity of the marginal labor, which uses machines of the old vintage, is unaltered until the labor reallocation process is completed. These facts cause the labor share to decline.

At the end of an adoption phase, all labor uses capital of the most recent vintage and the economy enters an innovation phase, during which capital of the last vintage still accumulates, its price decreases, the wage rate increases and output of the consumption good stagnates, generating an increasing labor share. Eventually the changes in relative prices make it profitable to innovate by turning the extra productive capacity into machines of a new vintage, embodying a new and better technology. At this point the cycle starts all over again.

[4]Because each new labor saving technology is embodied in *plants/machines* of a new *vintage*, all these words are used interchangeably across the paper.

The main contribution of the paper is to provide a dynamic general equilibrium theory of factor shares oscillations, as driven by endogenous labor saving technological progress, that is consistent with the main stylized facts of medium-run cycles.

Related Literature Motivated by labor share decline observed in recent decades (Elsby et al., 2013; Karabarbounis and Neiman, 2014), a few papers investigated issues similar to those we address. Acemoglu and Restrepo (2018) offers a model of endogenous innovation in which the labor share is constant at the long run balanced growth path, though it fluctuates during the transition to it. A shock to the automation technology initially pushes down wages and labor share, which discourages further automation efforts and creates incentives to create new labor intensive tasks and stabilize the labor share. Growiec et al. (2018) also contains a model of directed technological change with a similar intuition. Our model shares with these papers the view that factor shares are endogenously altered by the innovation/investment activity but differs in that the oscillations are endogenously persistent rather than a temporary response to exogenous shocks.

In our model, the labor share decreases during an adoption phase as labor reallocates from firms that use less advanced technology and have a larger labor share to those with more advanced technology and a smaller labor share. This view echoes the recent empirical literature on the labor share decline, which emphasizes the role of technology. Some noteworthy papers are (Autor et al., 2020; Dinlersoz and Wolf, 2018; Hubmer and Restrepo, 2021; Martinez, 2021; Kehrig and Vincent, 2021; Boldrin and Zhu, 2021).

A number of papers have studied the fluctuations of factor shares at the business cycles frequency (Boldrin and Horvath, 1995; Gomme and Greenwood, 1995; Young, 2004; Boldrin and Fernandez-Villaverde, 2006; Ríos-Rull and Santaeulalia-LIopis, 2010; Choi and Rios-Rull, 2019). Models in this literature typically rely on the combination of exogenous shocks and non-competitive wage setting to generate a counter-cyclical movement of the labor share. In Leon-Ledesma and Satchi (2019), upon a negative technology shock, the labor share rises under capital labor complementarity in the short run, but declines over the medium run as firms overcome the adjustment cost and switch to more labor saving technologies.

As mentioned, few papers have investigated the medium run behavior of factor shares. Growiec et al. (2018) decomposes the fluctuation of labor share into short (≤ 8 years), medium ($8 - 50$ years) and long run (≥ 50 years) frequencies, and find that medium-to-long run fluctuations accounts for about 80% of total labor share fluctuations. Inspired by the European experience in the 1970s and 1980s, Blanchard et al. (1997) and Caballero and Hammour (1998) have explored the factor share dynamics over the medium-run induced by exogenous changes in real wages. In these papers the variation in real wages and, hence, in labor share is exogenous and the subsequent response of firms leads back to the initial steady state equilibrium.

Finally, we note the similarities between some points of our model and the literature on directed technological change(Acemoglu, 2002). The main differences are that (i) we claim growth cycles are 'caused' by labor-saving technological change, and (ii) we focus on the fundamental bias (labor vs capital) in a perfectly competitive environment. Finally, this paper is also related to the vast and relatively forgotten literature on endogenous cycles. Among the many notable papers that could be quoted the one closer to our intuition, at least in spirit, is Goodwin (1967), though there is no growth, either exogenous or endogenous, in that model.

The rest of the paper is organized as follows: Section 2 presents the stylized facts. Section 3 outlines the basic model with exogenous labor supply and characterizes the competitive equilibrium, while section 4 studies the implications of endogenous labor supply and exogenous population growth. Section 5 provides concluding remarks. Most proofs and calculations are in the appendices.

2 Stylized Facts

Our crucial finding is that factor shares oscillate regularly at medium-run frequencies and that they display the following correlations:

- The growth rates of the labor share and of labor productivity are negatively correlated.
- The growth rates of the labor share and of real wages are positively correlated.
- The growth rate of the labor share is positively correlated with those of employment and of hours worked.
- The growth rates of the labor share and value added are weakly negatively correlated.

2.1 Calculation of the Factor Shares

To calculate the labor income share for the whole economy, we follow the standard approach of dividing proprietor's income between labor and capital according to the factor shares observed in the rest of economy (Cooley and Prescott, 1995). We define the gross labor share as

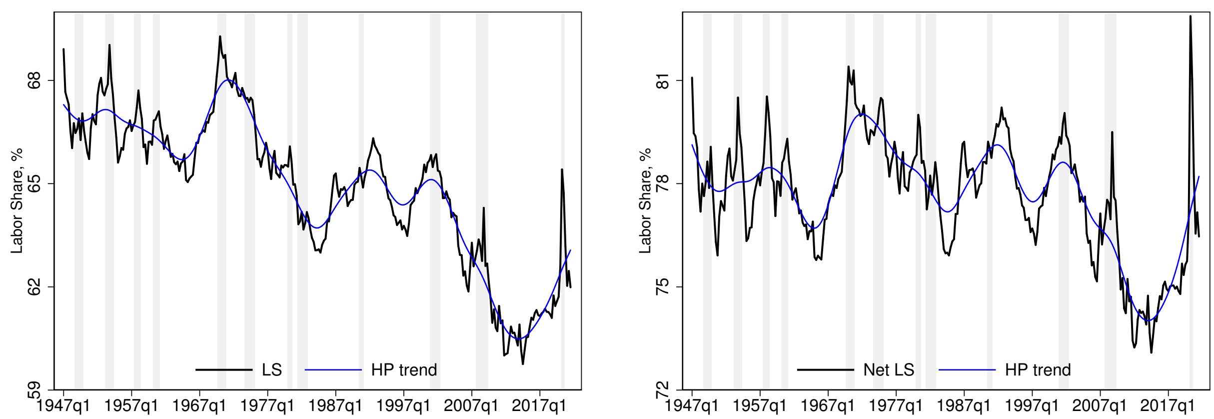
$$LS = \frac{\text{Compensation of Employees}}{\text{National Income} + \text{Depreciation} - \text{Proprietors Income} - \text{Tax}},$$

where "Tax" stands for taxes minus subsidies on production and imports.[5] The net labor share is defined as

$$LS^{net} = \frac{\text{Compensation of Employees}}{\text{National Income} - \text{Proprietors Income} - \text{Tax}}.$$

2.2 Labor Share

By applying these definitions to the US data we calculate the US quarterly gross and net labor share series for 1947Q1-2021Q3, which are shown next.



(a) Gross Labor Share

(b) Net Labor Share

Figure 2.1: Gross and Net Labor Share, 1947Q1-2021Q3

Note: This figure plots the gross and net labor income share in GDP, at the quarterly frequency, from 1947Q1 to 2021Q3. Also plotted are the HP trend with a smoothing parameter of 1600 and the NBER dating of recessions.

Regular medium-run fluctuations are visible in the behavior of the HP trend. The latter is a reasonable summary of the 2 to 4 years moving averages which define the *medium-run* and we study in detail in the next section. With the expression "medium run fluctuations", we refer to the fairly regular cyclical movements of the labor share visible in the behavior of its HP trend.[6]

[5] Data used in calculating the aggregate labor share is taken directly from NIPA, Table 1.7.5 'Relation of gross domestic product, gross national product, net national product, national income, and personal income', and Table 1.12 'National income by type of income', covering the period from 1947-q1 to 2021-q3.

[6] We use the terminology 'medium-run fluctuations' and 'medium-run cycles/ dynamics' interchangeably. They have been often used, in the literature, with somewhat different meanings. In Comin and Gertler (2006), they denote

Historically, it oscillated within a range of about 6 percentage points of GDP between 1947 and about 2000, after which a pronounced decline ensued that lasted until 2014. It then displayed a strong rebound which lasted until the 2020 recession. In the HP trend a full cycle lasts, typically, about 10-20 years, coincident with some, but longer than, average business cycles.[7]

Similar patterns hold if we extend the labor share series to the 1929-1946 period, as shown in Figure A.1. Koh et al. (2021) document that NIPA's capitalization of intellectual property products (IPP) accounts for most of the observed labor share decline—the labor share under the old NIPA classification is trend-less. We replicate their exercise in Figure A.2 and confirm that from 1929-2020, though the long run trend is altered, the medium run behavior of the labor share is not.[8]

Crucially for our research, note that: (i) the medium-run cycles before WWII are essentially identical to those after it; (ii) the net labor share exhibits fluctuations around a century-long flat trend.

2.3 Capital Share

Consider next the complement to the labor share, the capital share. While, in the national income and product accounts, labor income consists of a single item (compensation of employees), capital income is the sum of four different components: *rental income of persons*, *corporate profits*, *net interest and miscellaneous payments*, and *consumption of fixed capital*. They are plotted in Figure 2.2.

Corporate profits account for most of the cyclical pattern, while net interest is relatively a-cyclical, as is rental income. Depreciation typically peaks in recessions but this does not alter the overall cyclical properties of gross factor shares. Given that the short and medium-run dynamics of the capital share in the US is mainly driven by corporate profits, in the next section, we focus on data from the Non-Financial Corporate Sector. This is consistent with the theoretical framework proposed later, which has nothing to say about the functional distribution of income in the Public and in the Financial sectors. The factor share cycles documented for the whole economy

deviations from trend over frequencies longer than business cycles'. Our use is akin to that of Blanchard et al. (1997). We have tried to follow Comin and Gertler (2006) and extract the medium run cycles using a Baxter-King band pass filter with a lower and upper limit frequency of 2 and 200 quarters. The obtained medium run cycles coincide well with the HP trend.

[7] From Figure 2.1, in US post-WWII period the net labor share contains the following decreasing-then-increasing cycles: 1947Q1-1958Q3 (11.5 years), 1958Q3-1972Q1 (13.5 years), 1972Q1-1991Q4 (19.75 years), 1991Q4-2001Q3 (9.75 years), 2001Q3-present (20.25 years)

[8] IPP Data is only available at the annual frequency. We downloaded data from NIPA in March 2022 when IPP values for 2021 have not yet been released.

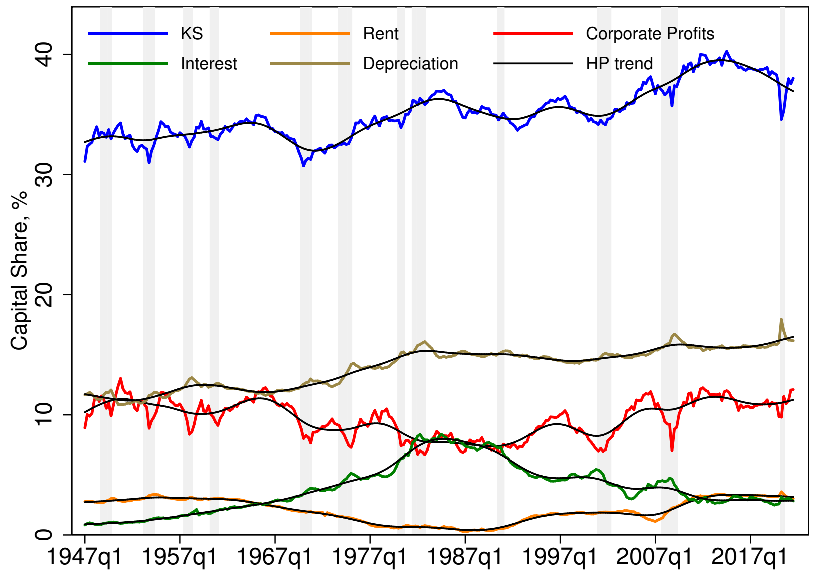


Figure 2.2: Capital Share and Its Components

Note: This figure plots the share of capital income and of its components, at the quarterly frequency. Also plotted are the HP trends with a smoothing parameter of 1600 and the NBER dating of recessions.

hold even more strongly for the non-farm business sectors and the non-financial corporations sector, which are closer to the theoretical object analyzed in our model.[9] The middle panel of Figure A.3 in the appendix show that the labor share in the non-farm business sector drives the dynamics of the one for the whole economy, while the bottom panel confirms that the labor income share in the non-financial corporate sector, which suffers the least from measurement issues, displays the same cyclical pattern.

2.4 Medium-Term Correlations

Next we examine the correlation between factor shares and other aggregate variables in the medium run. To do this we smooth out business cycle fluctuations by using the H-P trend with a smoothing parameter $\lambda = 1600$ as well as calculating moving averages (MA) over 9 quarters

[9]For the non-farm business sector (NFBS), we calculate the labor share using the same method as for the whole economy. We obtain "Tax minus subsidies on production and imports" for NFBS by subtracting the farming portion from that for the whole economy. As documented in Elsby et al. (2013), the treatment of factor shares for proprietors' income affects the magnitude of the labor share decline after the 1980s. The medium run fluctuations are, however, robust to such treatment. For the non-financial corporate sector (NFCS), BLS provides data on labor share, labor productivity, employment and working hours, which we utilize later in this section. The LS for NFCS, as provided by BLS, is computed as compensation of employees divided by value added. When this is adjusted for "Tax minus subsidies on production and imports" its level increases of about 6 percentage points, but the dynamics is essentially the same.

(2-year), 13 quarters (3-year) and 17 quarters (4-year). From now onward these are our labor (LS) and capital (KS) shares.

The LS co-moves systematically with other aggregate variables. Table 2.1 presents the correlation between the growth rates of the LS, labor productivity (LP), wage rate (WAGE, i.e. real compensation per hour worked), employment (EMP), and real value added (VADD) at different frequencies.[10] LS is already defined in percentage term, we calculate the difference between t and $t+1$ as its growth rate. For all other labor market variables, the growth rate is defined as the logarithmic change.

Table 2.1: Sample Correlations

Variable	LP	WAGE	EMP	VADD
Quarterly	-0.52^{***}	0.41^{***}	-0.35^{***}	-0.61^{***}
9-Quarter MA	-0.44^{***}	0.35^{***}	0.22^{***}	-0.13^{**}
13-Quarter MA	-0.36^{***}	0.40^{***}	0.27^{***}	-0.06
17-Quarter MA	-0.24^{***}	0.48^{***}	0.32^{***}	0.05
HP trend	-0.22^{***}	0.40^{**}	0.35^{***}	0.08

Note: This table gives the correlation between growth rate in LS and in other labor market variables at various frequencies. $^{***}: p < 1\%$; $^{**}: p < 5\%$; $^{*}: p < 10\%$. Data is for the Non-Financial Corporate Sector in the US, from 1947Q1-2021Q3.

At all frequencies, there is a significantly negative correlation between LS and LP, and a significantly positive correlation between LS and WAGE. Panel (a) of Figure A.4 plots the growth rate of the 3-year moving averages of LS and LP while panel (b) plots those for LS and WAGE.

The correlation coefficient between LS and EMP changes sign as we move from business cycle to lower frequencies. The correlation is significantly negative for quarterly data but it becomes positive afterwards, which suggests that the medium run dynamics differ qualitatively from business cycle fluctuations. Panel (a) of Figure A.5 plots the growth rates of the 3-year moving averages of LS and EMP. The correlation between the LS and working hours is qualitatively and quantitatively similar to the correlation between LS and EMP at all frequencies, as shown by

[10]The Bureau of Labor Statistics provides data on LS, value added, labor productivity, employment and hours worked for the non-financial corporate sector. LS is defined as the share of "compensation of employees" in "value added" for the non-financial corporate sector. The data source for LS is "U.S. Bureau of Labor Statistics, Nonfinancial Corporations Sector: Labor Share for Employees [PRS88003173], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/PRS88003173".

Table A.1 in the appendix.

The LS and VADD have a significantly negative correlation at the quarterly frequency. It becomes less negative for 2-year moving averages and statistically insignificant at lower frequencies. Panel (b) of Figure A.5 plots the growth in the 3-year moving averages of LS and VADD.[11]

To check that the correlations reported in Table 2.1 are not driven by the long-run trends or by short-run recessions, we regress the growth rates of the LS (quarterly series, moving averages, and HP trend) against the growth rate of other labor market variables, controlling for a linear time trend and a recession dummy. The results are presented in Tables A.2-A.4. The correlation between the growth rates of the LS and labor productivity, wage, and employment is not affected. The correlation between the growth rates of LS and value added, is significantly negative for quarterly series, 2- and 3- year moving averages; for 4-year moving averages and the HP trend, the correlation becomes insignificant, though still negative.[12] In Table A.5, we report the correlation coefficients for the Non-Farm Business Sector of the United States, and find a pattern that is very similar to the Non-Financial Corporate Sector.

In appendix B, we further confirm that the LS displays similar medium rum fluctuations, and that the same correlations also hold true for other OECD countries.

To summarize, in the medium run there is a robust negative correlation between changes in the LS and changes in labor productivity, and a robust positive correlation between changes in the LS and changes in wages, employment and working hours. In general, changes in the LS and in value added are negatively correlated but such correlation becomes weaker as the time frequency gets lower.

In the next two sections we build a model of endogenous growth replicating these four sets of stylized facts.

[11]The pattern for the 2- and 4- year moving averages and for the HP trend are similar to the 3-year moving average.

[12]In Table A.4 we use data for 1947Q1-2021Q3. The correlation between the LS and the HP trend of real value added is positive and significant at the 10% level. In an earlier version, using data up to 2020Q3, the same correlation was negative and insignificant. As this point estimate seems rather unstable, we rely more on estimates obtained using the moving average terms. Arguably, they impose less assumptions on the raw data, and consistently deliver a weakly negative correlation between LS and value added.

3 The Model

Our model rests on two assumptions: (i) capital and labor are complementary inputs;[13] (ii) technical progress is labor saving and embodied in capital goods.[14] Everything else is standard: recursively complete markets over an infinite horizon and a representative agent with perfect foresight receiving utility from consumption and leisure. We consider first the case of exogenous labor supply.

Preferences The representative household maximizes the following utility over the infinite horizon,

$$\max \int_0^\infty e^{-\rho t} \log c(t) dt.$$

where $c(t) = \sum_{j=0}^{\infty} c_j(t)$, with $c_j(t)$ the consumption flow from technology j at instant t. The household inelastically supplies one unit of labor.

Production Production takes places in three different sectors denoted by $s = 1,2,3$. Each sector is composed by a continuum of identical firms endowed with capital of some vintage.[15] The first sector produces the consumption good, the second investment goods and the third a new vintage of capital embodying a better technology.

Technological Vintages There exists a countably infinite number of potential technologies, indexed by the subscript $j = 0,1,....$. Technologies are embodied in capital goods, hence $k_j^s(t)$ denotes the stock of capital embodying technology j installed in sector s at time t. We say that a technology j is active in sector s at time t if $k_j^s(t) > 0$.[16]

Technological Progress A technology with index j is better than a technology with index $j' < j$ for two reasons. First, to produce one unit of consumption, a unit of capital j requires less labor than a unit of capital j', i.e. technological progress is labor saving. Second, technological progress is incremental insofar as capital $j+1$ can be obtained, at a cost, only from capital j and

[13] That capital and labor are complementary inputs in the aggregate is supported by most empirical estimates. See e.g. Antras (2004), Klump et al. (2007), Oberfield and Raval (2021).

[14] Research on biased technological progress dates back to at least Hicks (1932) and Kennedy (1964). Recent contributions include Acemoglu (2002) and Jones (2005), among others. Manuelli and Seshadri (2014) and Acemoglu and Restrepo (2021) study two different historical examples of labor-saving technological change driven by variations in the cost of labor.

[15] Because firms are identical in each sector, we will talk, indifferently, either of a representative firm with a stock of capital equal to $k^s(t)$ or of a continuum of identical firms, each one with $k^s(t)$ units of capital.

[16] Think of them as plants with constant returns to scale.

not from any other $j' < j$.

Consumption Sector The first sector produces consumption, $c_j(t)$, using capital $k_j^1(t)$ and labor $\ell(t)$ according to a fixed coefficient production function,

$$c_j(t) = \min\{k_j^1(t), \gamma^j \ell(t)\}, \quad \gamma > 1.$$

This means that, for every technology j, capital and labor are perfectly complementary inputs.[17] The assumption that γ is greater than one captures the fact that technological progress is labor saving. As a new vintage is adopted, the labor-input requirement to produce 1 unit of consumption decreases by a factor $1/\gamma$.

Investment Sector The second sector produces additional units of capital of type j from capital of the same vintage, according to

$$\dot{k}_j(t) = bk_j^2(t), \quad b > 0.$$

The investment sector allows every kind of capital to self-accumulate at the rate b *after* it has been introduced.

Innovation Sector The third sector innovates by producing a new vintage of capital, $j+1$, from capital of vintage j

$$k_{j+1}(t) = \frac{k_j^3(t)}{a}, \quad a > 1.$$

Capital stock of type j used in the innovation sector is transformed instantaneously into the new kind of capital $j+1$. This can be obtained directly only from type j and not from any $j' < j$. However, capital $j+1$ can be obtained from capital $j', j' < j$ by applying the innovation technology $j - j' + 1$ times. The innovation ratio, in this case, would be equal to $a^{j'-j-1}$

Because capital j can be employed in any of the three sectors, at any t the following resource constraint holds

$$k_j(t) = k_j^1(t) + k_j^2(t) + k_j^3(t).$$

[17] A Leontief production function is used to obtain analytical solutions. The qualitative results hold for a general CES production function with gross capital-labor complementarity. Similarly, we use the simple exponential form to model technological progress only in order to obtain explicit solutions.

The accumulation equation for capital j, therefore, is

$$dk_j(t) = bk_j^2(t)dt - k_j^3(t) + \frac{k_{j-1}^3(t)}{a}.$$

This equation says that the stock of capital j changes because of (i) self accumulation (first term), (ii) full depreciation of the amount used to innovate (second term) and, (iii) innovation from capital $j-1$ (third term). Note that we allow for discrete conversions from any vintage of capital to the next, as captured by the second and third terms.

This economy is an ordinary diminishing return economy with three sectors: consumption, investment and innovation. Diminishing returns to capital accumulation derives from the fact that capital and labor are complementary inputs and available labor is limited. As there is perfect competition, the welfare theorems hold and the efficient allocation can be decentralized as a competitive equilibrium and vice versa.[18] Therefore, the competitive equilibrium prices correspond to the co-state variables of the planner's problem, a fact we exploit repeatedly in characterizing the dynamic equilibrium.

Three parametric assumptions are crucial. (1) $b > \rho$, the rate of capital self-reproduction is larger than the discount rate, which makes accumulation profitable. (2) $a > 1$, innovating is costlier than investing in old capital stock, hence innovation will not take place until the introduction of a new vintage of capital becomes profitable. (3) $\gamma > 1$, i.e. machines of a more advanced vintage require less labor to produce one unit of the consumption good. Below we prove that, under these assumptions, the competitive equilibrium of the economy settles into a sequence of growth cycles. Each cycle contains an *adoption* phase, where capital goods of two consecutive vintages are simultaneously used to produce consumption and labor is being reallocated from the less to the more advanced vintage, and an *innovation* phase, where capital is accumulated till its price decreases enough to make it profitable to use a new vintage to produce the consumption good.

3.1 Stylized Properties of the Model

In our model, as in the data shown in Figure 2.1, the labor share oscillates regularly and recurrently during the growth cycles, as shown in Figure 3.1. In particular, it decreases during an

[18]The competitive equilibrium of the economy can be described in the usual way by combining utility and profits maximization.

adoption phase, and increases during an innovation phase.

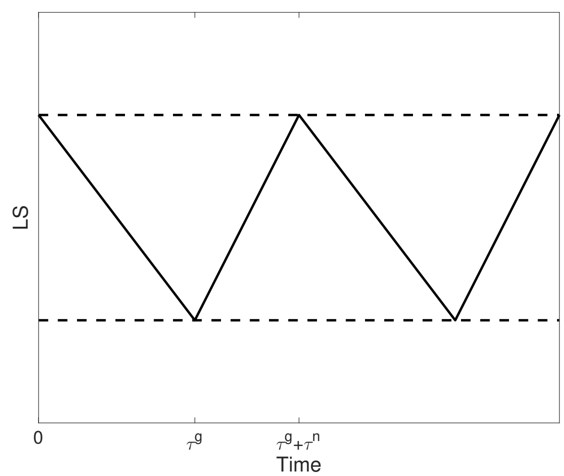


Figure 3.1: LS Dynamics in the Model

Note: This figure illustrates the endogenous and recurrent cycles of LS in the model. τ^g and τ^n are the length of the adoption and innovation phases of a growth cycle.

The model also generates the correlation between labor share and the four macro variables summarized in the four bullet points at the start of Section 2 and displayed in Table 2.1. The first two are discussed next.

3.1.1 The growth rates of the labor share and of labor productivity are negatively correlated

The cycles of our model are composed of two phases: adoption and innovation. During the adoption phase wages are constant but labor productivity increases whereas during the innovation phase the wage increases but labor productivity stagnates. Therein the negative correlation between LS and labor productivity.

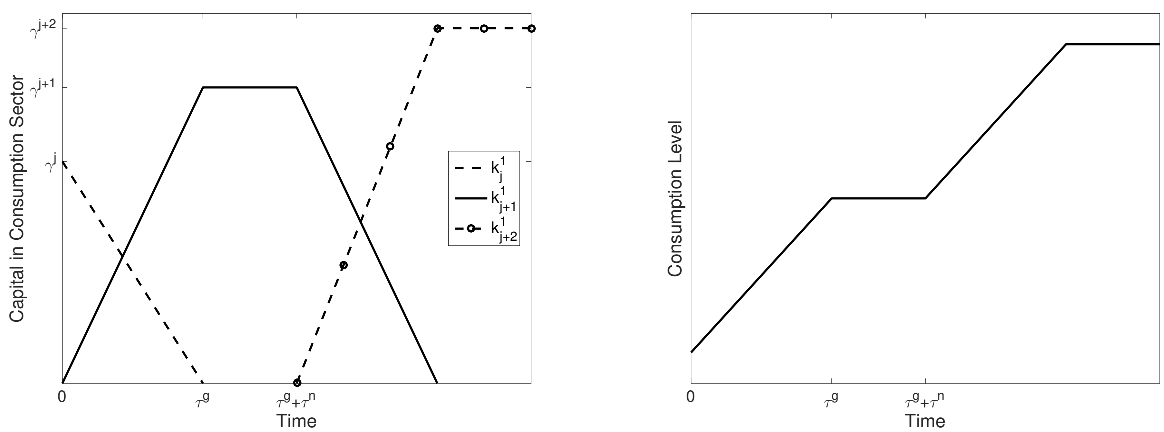
3.1.2 The growth rates of the labor share and of real wages are positively correlated

During the adoption phase wages are constant but productivity and output of the consumption good increase, hence the labor share of total value added decreases. On the contrary, during the innovation phase productivity is constant, and so is output of the consumption sector, while wages increase. This yields the positive correlation between LS and wage.

The correlations summarized in the third and fourth bullet points, and in the last two columns of Table 2.1 require a variable labor supply and the computation of total output for the three sectors. They are discussed in section 4.

3.2 Description of the growth cycles

The left panel of Figure 3.2 illustrates the evolution of capital in the consumption sector for two consecutive cycles.[19] Start at time $t = 0$, when vintage $j + 1$ is used for the first time to produce the consumption good that, until then, was entirely produced by capital of vintage j. Consumption increases over time (see the right panel) as plants of vintage j are replaced by those of vintage $j + 1$ and labor reallocates from the former to the latter. This phase ends at time $t = \tau^g$, at which point capital of vintage $j + 1$ employs all labor. As shown later, at $t = \tau^g$ capital $j + 2$ will not be immediately introduced. It will be "invented" and accumulated until its price is low enough to make its adoption profitable. This takes place at $t = \tau^g + \tau^n$.[20] From $t = \tau^n$ to $t = \tau^g + \tau^n$ output of the consumption sector is constant as labor is already fully employed by capital $j + 1$. The recurring growing-then-stagnant evolution of the consumption sector is illustrated in the right panel of Figure 3.2.



(a) Capital in the Cons. Sector

(b) Consumption

Figure 3.2: Capital Input and Output in the Consumption Sector

Note: The left panel illustrates the evolution of capital used in the consumption sector; the right panel reports the output of consumption. γ^j is the amount of capital j when it employs all labor. τ^g and τ^n are the length of an adoption and innovation phase, respectively.

When, during the *adoption phase*, vintages j and $j + 1$ are simultaneously used to produce the final good, factor income shares change as adoption progresses. When technology j and $j + 1$

[19] What is plotted here is an illustration of the general pattern. The actual dynamics of the stock of capital of a specific vintage is generally nonlinear in time and will be derived later.

[20] At $t = \tau^g + \tau^n$, the amount of labor employed by capital $j + 1$ is 1 and that by capital $j + 2$ is 0.

are simultaneously used we have

$$LS_j = \frac{a-1}{a-1/\gamma}; \quad LS_{j+1} = \frac{1}{\gamma}\frac{a-1}{a-1/\gamma}.$$

As $\gamma > 1$, firms utilizing technology $j+1$ admit a smaller LS than firms using j and the labor reallocation process reduces the aggregate LS. This is because aggregate labor productivity increases but wages, determined by the marginal worker employed in vintage j, do not. The key step occurs once the adoption phase ends: all labor is employed with machines of type $j+1$ and productivity growth ceases.

Is there a reason to keep accumulating capital of vintage $j+1$? We show in subsection 3.3. that there is: as its stock accumulates its price drops until it becomes profitable to use the innovation sector to turn machines of type $j+1$ into machines of type $j+2$, thereby introducing a new technology. This is the *innovation phase* during which the wage rate and the LS increase while labor productivity remains constant.

In section 3.3. we also show, formally, that the aggregate LS decreases from $\frac{a-1}{a-1/\gamma}$ to $\frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$ during the adoption phase and increases back to exactly $\frac{a-1}{a-1/\gamma}$ during the innovation phase. Figure 3.1 illustrates the recurring LS cycles.

Figure 3.3 depicts the stylized dynamics of the successive vintages of capital stock in the whole economy. At time $t = 0$ the transition from vintage j to vintage $j+1$ starts, which ends at τ^g. At this time all workers are employed with machines of type $j+1$, of which there are $\gamma^{j+1} > \gamma^j$ units. For τ^n units of time nothing changes in the consumption sector while, in the investment sector, capital of vintage $j+1$ is accumulated further. This "over-accumulation" of capital continues until the price of capital reaches a level low enough for the adoption of capital $j+2$ to become profitable.[21] The new adoption phase begins at $t = \tau^g + \tau^n$ and the growth cycle starts again.

3.3 Characterization of the Competitive Equilibrium

We now formally derive the properties of the competitive equilibrium. Use marginal utility as the numeraire, hence the price of consumption at t is $1/c(t)$. Denote with $q_j(t)$ the price of capital j at time t. The physical rate of return in the investment sector is b. Zero profit implies that b

[21] As we show is subsection 3.3, invention of capital $j+2$ may occur at any point during the innovation phase. Paths with different timing of invention are equivalent in that capital $j+2$ starts producing consumption always at the same moment and the same amount of consumption is obtained at any point in time.

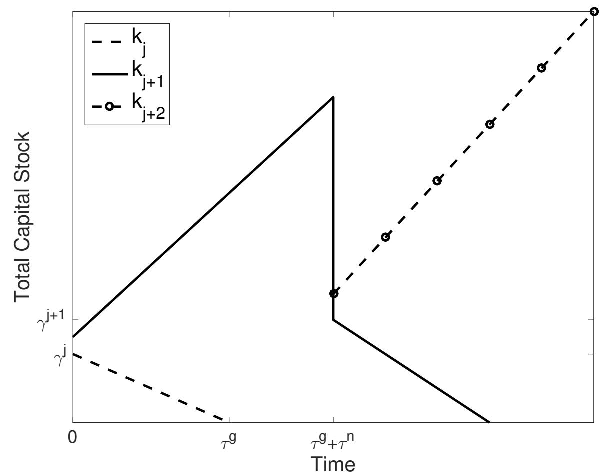


Figure 3.3: Evolution of Total Capital Stock

Note: This figure plots the evolution of aggregate capital stock of different vintages over time in the model.γ^j is the amount of capital j when it employs all labor. τ^g and τ^n are the length of an adoption and innovation phase, respectively.

plus capital gains must equal the subjective discount rate

$$b + \dot{q}_j(t)/q_j(t) = \rho,$$

or equivalently,

$$\dot{q}_j(t)/q_j(t) = -(b-\rho) < 0.$$

As it accumulates, the price of capital of type j decreases over time. Its level is characterized in the following proposition.

Proposition 1: *No more than two vintages of capital are simultaneously used to produce consumption, and these must be consecutive vintages. If j' is used to produce consumption, the price of capital $j, j > j'$ satisfies*

$$q_j(t) \geq v_j(t) \equiv \frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}} \frac{1}{bc(t)}$$

with equality if j is also used to produce consumption.

Proof: see Appendix.

The key step in the proof is the computation of the price of capital. Without loss of generality, assume both capital j' and j, $j > j'$, are used to produce the consumption good. Denote with w, r_j and $r_{j'}$, respectively, the wage rate, and the return to capital j and to capital j', all in units of the consumption good.[22] The zero profit condition in the consumption sector implies

$$1 - r_j - \frac{w}{\gamma^j} = 0, \quad 1 - r_{j'} - \frac{w}{\gamma^{j'}} = 0,$$

and the zero profit condition in the innovation sector leads to[23]

$$r_j = a^{j-j'} r_{j'}.$$

This is a system of three independent equations with three unknowns. It yields

$$w = \gamma^{j'} \frac{a^{j-j'} - 1}{a^{j-j'} - 1/\gamma^{j-j'}}; \quad r_j = \frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}},$$

and $r_{j'} = r_j / a^{j-j'}$. The rental rate divided by b, the rate of capital accumulation, gives the value of the stock of capital, which is

$$v_j(t) = \frac{1}{c(t)} \frac{r_j(t)}{b} = \frac{1}{bc(t)} \frac{\gamma^{j'-j} - 1}{\gamma^{j-j'} - 1/a^{j-j'}}.$$

The $1/c(t)$ term converts the price of capital in units of the numeraire, which is marginal utility.

When both capital j and $j', j > j'$, are used in production, one extra unit of capital j used in producing the consumption good demands $1/\gamma^j$ units of labor, which leads to unemployment of $\gamma^{j'-j}$ units of capital j' for a given level of labor supply. The value of one extra unit of capital j should therefore compensate for the units of capital j' that become obsolete. This is the reason why the coefficient in the formula of Proposition 1, $\frac{\gamma^{j-j'}-1}{\gamma^{j-j'}-1/a^{j-j'}}$, is less than 1. This "replacement effect" explains why there are at most two consecutive vintages of capital simultaneously used in production. When capital j' is used to produce the consumption good, zero profit in the innovation sector implies that the price of capital goods of type $j > j' + 1$ is higher by a factor of $a^{j-j'}$. However, the replacement effect shows that its value in the consumption sector is not as high. It is therefore not profitable to use any capital $j > j' + 1$, to produce consumption.

Proposition 2 characterizes the dynamic behavior of output and factor shares.

[22]For computational convenience, factor prices are expressed in units of the consumption good. Multiplying these prices by $1/c(t)$ gives the price in terms of marginal utility.

[23]The zero profit condition in the innovation sector is originally written in terms of capital prices. As shown in the appendix, there is a linear relation between the price and the rental rate of capital.

Proposition 2: *Consumption grows at the rate $b-\rho$ during an adoption phase, which lasts for $\tau^g = \frac{\log \gamma}{b-\rho}$ units of time. It is followed by an innovation phase, lasting $\tau^n = \frac{\log a}{b-\rho}$ units of time during which consumption remains constant. The total length of a cycle is*

$$\tau^* = \frac{\log a + \log \gamma}{b-\rho}$$

The LS declines from $\frac{a-1}{a-1/\gamma}$ to $\frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$ in the adoption phase, and increases back to $\frac{a-1}{a-1/\gamma}$ in the innovation phase.

Proof: see Appendix.

During the adoption phase consumption grows as labor is shifted from the less advanced capital j to the more advanced $j+1$. The labor shares in firms employing capital j and $j+1$ are

$$LS_j = \frac{wl_j}{\gamma^j l_j} = \frac{a-1}{a-1/\gamma}, \quad LS_{j+1} = \frac{wl_{j+1}}{\gamma^{j+1} l_{j+1}} = \frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$$

which are the formulas we presented earlier. Reallocation of labor from capital j to $j+1$, i.e. from firms with a greater LS to those with a smaller one, decreases the aggregate share of income going to labor. The aggregate LS declines from $\frac{a-1}{a-1/\gamma}$ to $\frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$ at the end of the adoption phase, when all labor works in firms using technology $j+1$.

3.4 Levels of capital in the initial phase

We have characterized the stable growth-cycle our model-economy converges to but, in doing so, we have abstracted from its initial conditions, which we consider next.

Denote with $j=0$ the least advanced type of capital and with τ_j the time at which capital j is first employed in producing the consumption good, hence with $k_j(\tau_j)$ the stock of capital of type j when it is first used in the production of the consumption good. Without loss of generality, start with an adoption phase when capital j and $j+1$ are simultaneously used. It turns out that $k_{j+2}(\tau_{j+2})$ and $k_{j+1}(\tau_{j+1})$ satisfy the following relation,

$$\frac{k_{j+2}(\tau_{j+2})}{\gamma^{j+1}} = (a\gamma)^{\frac{\rho}{b-\rho}} \frac{k_{j+1}(\tau_{j+1})}{\gamma^j} - x,$$

where $x \equiv a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}} - 1)\frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)} > 0$.[24] Figure 3.4 illustrates the normalized capital stock $\frac{k_{j+2}(\tau_{j+2})}{\gamma^{j+1}}$ as a function of $\frac{k_{j+1}(\tau_{j+1})}{\gamma^{j}}$. As $(a\gamma)^{\frac{\rho}{b-\rho}} > 1$, the function is steeper than the 45-degree line. There exists a unique steady state value for the normalized capital stock. An initial value below the steady state eventually leads to a negative capital stock and any initial value above it results in an explosion.[25]

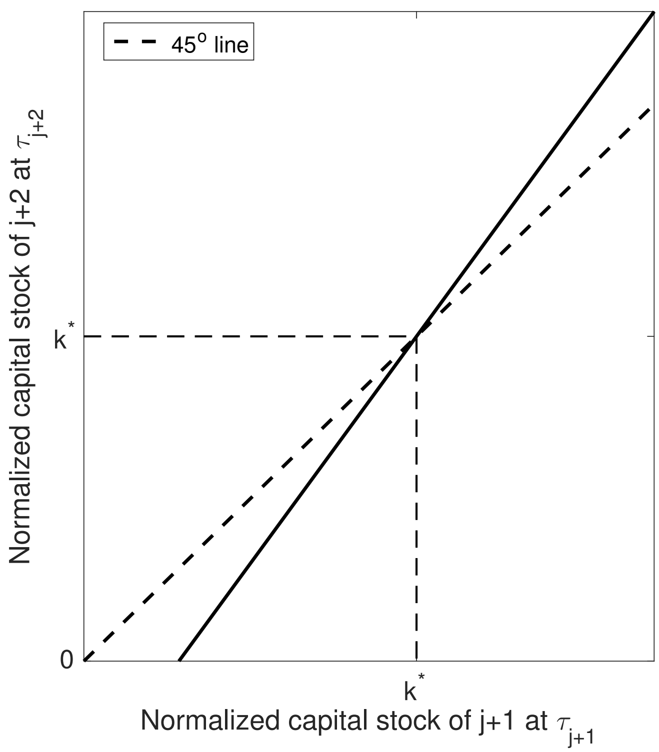


Figure 3.4: Steady State of Normalized Capital

Note: This figure illustrates the linear relationship between the amount of normalized capital stock k_{j+1} and k_{j+2}, evaluated at their first use in production, τ_{j+1} and τ_{j+2} respectively. Also plotted is the 45-degree line.

To investigate the initial capital allocation, we begin with the case $0 < k_0(0) < 1$, when the initial stock of capital 0 is not enough to employ all labor at $t = 0$. The first recurring cycle starts when machines of type 0 and of type 1 are simultaneously used in producing the final good. We use τ_1 to denote the endogenous starting time of this first cycle. The amount of capital 1 at $t = \tau_1$, $k_1(\tau_1)$, should then equal the steady state value calculated above. During the initial unemployment phase, capital 1 is too expensive to be introduced immediately: there is excess labor, no reason to innovate. The competitive equilibrium assigns a certain amount of capital 0 to sector

[24]See Appendix C.1 for the details of this calculation.
[25]Both these paths violate the transversality condition. See Appendix C.1 for details.

1, denoted as $k_0^1(0)$, and the rest, $k_0(0) - k_0^1(0)$, to sector 2. During this initial phase, $k_0^1(t)$ and $c(t)$ grow over time until full employment is reached at time $t = \tau_0^g$, when consumption equals 1, i.e. $c(\tau_0^g) = 1$.

The first innovation phase starts at this point: capital of type 0 is accumulated further and consumption remains constant. This phase ends at $t = \tau_1$ when the (implicit) price of capital 1 equals its value in production. It should be noted that the value of τ_1 is an endogenous variable as it depends on the initial stock of capital. At $t = \tau_1$, the economy enters the recurring cycles and behaves as described above.

Given $k_0^1(0)$, with $0 < k_0^1(0) < k_0(1)$, one can calculate τ_1 and the value of capital 0 for $0 < t \leq \tau_1$. As shown in Appendix, the condition that the amount of capital 1 at τ_1 equals the steady state value, i.e. $k_1(\tau_1) = k^*$, uniquely determines the initial capital allocation.

Recall that $k_{j+1}(\tau_{j+1})$ and $k_{j+1}(\tau_{j+1} + \tau^g)$ stand for, respectively, the amount of capital $j+1$ at the beginning and at the end of the adoption phase in which capital j and $j+1$ are simultaneously employed. When the initial amount of capital of type 0 is greater than 1, i.e. $k_0(0) \geq 1$, its allocation is determined as follows. If $k_0(0) \in [1, k_1(\tau_1) * a + 1)$, then 1 unit of capital of vintage 0 is used in producing the consumption good and the remaining is used for self-accumulation. If $k_0(0) \in [k_{j+1}(\tau_{j+1}) * a^{j+1} + \gamma^j * a^j, k_{j+1}(\tau_{j+1} + \tau^g) * a^{j+1})$ for some $j \geq 1$, then $k_0(0)$ is immediately converted into both capital j and $j+1$ and the economy jumps to the corresponding adoption phase[26]. Finally, if $k_0(0) \in [k_{j+1}(\tau_{j+1} + \tau^g) * a^{j+1}, k_{j+2}(\tau_{j+2}) * a^{j+2} + \gamma^{j+1} * a^{j+1})$ for some j, then a portion of $k_0(0)$ is converted into enough capital $j+1$ to produce γ^{j+1} units of the consumption good while the rest goes to self-accumulation and the economy starts from the corresponding innovation phase.

Proposition 3 summarizes these results.

Proposition 3: *Depending on the quantity of the initial stock of capital, there might be an initial phase when a single vintage of capital is employed and accumulated. After that initial phase, the economy settles into a recurring adoption and innovation cycle. The value of capital stock j when it is first introduced at*

[26] Any given amount of the initial type 0 capital maps into a point in some future growth cycle. This correspondence is used to determine how much $k_0(0)$ is converted to capital j and how much to $j+1$, as well as the initial allocation of the converted capital among the three sectors.

$t = \tau_j$ satisfies $k_j(\tau_j) = \gamma^{j-1}k^*$, where k^* is defined as

$$k^* = \frac{x}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1},$$

with $x \equiv a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}} - 1)\frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$.

Proof: see Appendix.

4 Endogenous Labor Supply and Population Growth

We extend the basic model in order to address the correlation between LS, employment and value added.

4.1 Endogenous labor supply

Relaxing the assumption that labor supply is fixed at one has three consequences:

- (i) during the innovation phase employment grows, hence consumption also grows, though at a lower rate than during the adoption phase;
- (ii) the relative length of the two phases changes while the total length of a cycle does not;
- (iii) the correlation between the LS and employment becomes positive while that with value added is weakly negative for some constellations of parameter values.

The representative household's problem is now

$$\int_0^\infty e^{-\rho t}[\log c(t) - \zeta\frac{\eta-1}{\eta}\ell(t)^{\frac{\eta}{\eta-1}}]dt,$$

where $\zeta > 0$ and $\eta > 1$.

4.2 Correlation of LS and Hours Worked

To derive the dynamics of employment note that the first order condition w.r.t. working hours $\ell(t)$ is

$$\frac{w(t)}{c(t)} = \zeta * \ell(t)^{\frac{1}{\eta-1}}.$$

During an adoption phase the wage rate, determined by the zero profit conditions in sectors 1 and 2, is constant. As consumption grows at the rate $b - \rho$, the first order condition for labor supply implies that employment shrinks at the rate $(\eta - 1)(b - \rho)$.

In the innovation phase, consumption will not remain constant as a rising wage induces a higher labor supply, which increases output of the consumption good. Consider the innovation phase when only capital $j+1$ is used in consumption. Substitute the production function, $c(t) = \gamma^{j+1} l(t)$, into the first order condition for working hours to obtain $w(t) = \zeta \gamma^j \ell(t)^{\frac{\eta}{\eta-1}}$. Hence, during the innovation phase, employment grows at the rate $(\eta-1)/\eta$ times the growth rate of wages. Because the latter grow during this phase, employment and consumption also increase. Formally, we have the following proposition[27]

Proposition 4: *The economy with endogenous labor supply settles into a recurring cycle, consisting of an adoption phase, when consumption grows at the rate $b - \rho$, and an innovation phase, when consumption grows at the rate $\frac{\frac{\eta-1}{\eta}\log\gamma}{\log a + \frac{\eta-1}{\eta}\log\gamma}(b-\rho)$. The adoption phase lasts for $\tilde{\tau}^g = \frac{\log\gamma}{\eta(b-\rho)}$, and is followed by an innovation phase lasting $\tilde{\tau}^n = \frac{\log a + \frac{\eta-1}{\eta}\log\gamma}{b-\rho}$. The total length of a cycle is*

$$\tilde{\tau}^* = \frac{\log a + \log\gamma}{b - \rho}.$$

Further, both the LS and the level of employment decline in the adoption phase while they increase during the innovation phase.

Proof: see Appendix.

This establishes the third in the list of the stylized facts of Section 2.

4.3 Correlation of LS with Output Growth

Until now we have focused on the consumption sector because, in our model, the other two sectors have a LS equal to 0 by assumption. Total output of the model economy, though, consists of both consumption and investment, of which there are two kinds.

As we show in Appendix C.4, the growth rate of total output, in both the baseline and extended models, varies during the two phases of the growth cycle and we cannot prove that it is always higher in one of them. Depending on parameter values, total output may, therefore,

[27] We refer to Appendix C.3 for the details of the proof.

be either positively or negatively correlated with the LS, or not correlated at all. In the data we have found a negative correlation at the business cycles frequencies, which becomes weaker but remains negative as the time frequency decreases. These estimates are similar to the findings of Leon-Ledesma and Satchi (2019) and we find them in general agreement with the predictions of our model.[28] In the appendix, we derive parameter restrictions under which total output growth is negatively correlated with variations in the LS.[29]

This establishes the fourth and last stylized facts: LS and output growth are weakly negatively correlated at medium run frequencies.

4.4 Quantitative Performance

In both the baseline and extended models, consumption in the adoption phase grows at the rate $b-\rho$; The length of the cycles is determined by a, γ and $b-\rho$. The value of γ and a further determines the magnitude of the LS decline in an adoption phase and of LS increase in an innovation phase. Both values are independent of the capital vintage j by assumption in our model. Allowing these values to vary with j, the model generates cycles of different magnitudes and lengths, as well as non-symmetric cycles, e.g. a prolonged and pronounced LS decline followed by a short and small recovery.

We take the consumption growth rate, $b-\rho$, at 2% per year, recover the value of γ_j from the observed decline in LS, and set a to target a peak LS level of two thirds over a full cycle. Under this parameterization, a decline of the LS by 2-3 percentage points implies a length of 5-7 years for the whole cycle. A more pronounced LS decline, of about 5 percentage points like during 2000-2014, leads to a cycle of more than 10 years. These numbers align quite well with the data.[30]

[28] In our exercise, we report a negative (and insignificant at lower frequency) correlation between the medium run labor share and the medium run value added, which differs from Growiec et al. (2018) who finds that the medium run component of labor share, extracted from frequency domain analysis, varies positively with output (not medium run output). Another reason that might cause the difference is that we study the non-financial corporate sector, while Growiec et al. (2018) focuses on the aggregate economy. When we use data for the aggregate economy, we also find a positive correlation between the medium run labor share and the medium run GDP.

[29] The baseline model also predicts a counterfactual negative correlation between consumption and investment because during the innovation phase the latter grows while the former does not. In Appendix C.4 we show that this is not the case in the model with endogenous labor supply.

[30] We use a logarithmic utility function in the baseline model. With a general CRRA utility function, $u(c) = \frac{c^{1-\theta}}{1-\theta}$, the price of consumption in terms of marginal utility is $c^{-\theta}$. One can calculate that the length of a cycle is equal to $\frac{log\gamma + 1/\theta * \log a}{(b-\rho)/\theta}$. Given a fixed consumption growth rate, $(b-\rho)/\theta$, the length of a cycle increases with the intertemporal elasticity of substitution, $1/\theta$.

4.5 Population growth

Population growth can also be incorporated into the model without loss of tractability. Assume the size of the representative household grows at an exogenous rate of n, with $n < \rho$. The household's effective discount rate becomes $\rho - n$. During an adoption phase, aggregate consumption grows at the rate $b - (\rho - n)$, while consumption per-capita still grows at the rate $b - \rho$. Therefore the length of the growth phase remains unchanged. During the innovation phase, aggregate consumption grows at the rate n, and the price of capital, in units of current marginal utility, decreases at the rate $b - (\rho - n)$. Therefore, the length of the innovation phase also remains the same. In an economy with population growth the per-capita variables behave exactly the same as in the baseline economy. The aggregate variables, though, align better with the four stylized facts considered above.

5 Conclusions

Since the end of WWII the factor shares of US national income, and of other advanced market economies, have displayed repeated and relatively regular cycles. While such oscillations have been of differing amplitude they are persistent over time and characterized by a few, robust, properties: the LS decreases when labor productivity increases faster than on average while, during the same periods, wage and employment grow less than on average. The opposite is true when the LS increases. Furthermore, factor shares display clear and stable correlations with employment and output. We have presented a theoretical model generating exactly such cycles in an endogenously growing economy.

At the core of our theory is the idea that technical progress is labor-saving and responds to relative factor prices. Technologies are embodied in capital goods, which can be accumulated over time, while the other production input (labor) cannot be augmented at a comparable speed. Accumulation of capital embodying a given technology increases wages, which provides incentives for creating a new, labor-saving, technology embodied in new machines. This interaction between factor prices and labor-saving technical progress generates perpetual medium run cycles along the equilibrium growth path.

To simplify an already cumbersome algebra, we have assumed in the model that labor is used only in the consumption sector. On the one hand, we doubt that the type of labor used to innovate or for creating new equipment is a particularly good substitute for the labor used in producing consumption goods, so we do not view the alternative assumption as especially realistic either. On the other hand, what happens if we require some sort of labor in the capital accumulation or innovation processes? The lengths of the two cyclical phases will be altered as the endogenous wage now becomes a factor in determining the cost of investing and/or innovating. However, this should not qualitatively affect the endogenous fluctuations in factor income shares around recurring growth cycles. During the growth phase factor prices are still determined by zero profit conditions, and the labor share decreases in the reallocation process from older to more advanced vintages of capitals. In the innovation phase, the accumulation of new capital would increase the wage rate and the share of labor in total income, as in the baseline model. Another simplification we have made in order to obtain analytical results is to adopt a Leontief production function, though a gross complementarity between capital and labor is sufficient for the mechanism in the model to work.

Our model focuses on medium run dynamics abstracting from shocks and propagation mechanisms that are relevant at the business cycle frequencies. The business cycle implications of our theory, in the presence of some kind of random disturbances occurring at a quarterly frequency, are left for future research. We also assume a single sector for the production of the final consumption good. Technology is expected to progress at different rates across industries in the real economy. Since at least Solow (1958), we are aware that, at the sector level, factor shares are not as stable as in the aggregate. Recent literature (Boldrin and Zhu, 2021; Hubmer and Restrepo, 2021) documents that there is substantial heterogeneity in labor share trends across industries. These considerations are also left for future research.

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Appendix A Tables and Figures

Table A.1: Correlation between Growth Rate in LS and in Working Hours

Frequency	Quarterly	9-Quarter MA	13-Quarter MA	17-Quarter MA	HP trend
Coefficient	-0.39^{***}	0.12^{**}	0.16^{***}	0.23^{***}	0.31^{***}

Note: This table gives the correlation between growth rate in LS and in total working hours at various frequencies. $^{***}: p < 1\%$; $^{**}: p < 5\%$; $^{*}: p < 10\%$. Data is for the Non-financial corporate sector in the US, from 1947Q1-2021Q3.

Table A.2: Regression coefficients, US time series

	Dep. var.: ΔLS, quarterly			
	(1)	(2)	(3)	(4)
ΔLP	-0.54^{***}			
	(0.05)			
$\Delta Wage$		0.49^{***}		
		(0.06)		
ΔEmp			-0.33^{***}	
			(0.05)	
$\Delta Vadd$				-0.44^{***}
				(0.03)
Recession FE	Y	Y	Y	Y
Linear time trend	Y	Y	Y	Y
R^2	0.28	0.20	0.13	0.43
Obs.	298			

Note: Non-Financial Corporate Sector, 1947Q1-2021Q3. $^{***}: p < 1\%$. The numbers in the brackets are the standard errors.

Table A.3: Regression coefficients, US time series

	Dep. Var.: ΔLS, 2-Year MA				Dep. Var.: ΔLS, 3-Year MA			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
ΔLP	-0.50^{***}				-0.45^{***}			
	(0.05)				(0.06)			
$\Delta Wage$		0.39^{***}				0.42^{***}		
		(0.06)				(0.06)		
ΔEmp			0.14^{***}				0.15^{***}	
			(0.04)				(0.04)	
$\Delta Vadd$				-0.17^{***}				-0.12^{***}
				(0.04)				(0.04)
Recession FE	Y	Y	Y	Y	Y	Y	Y	Y
Linear time trend	Y	Y	Y	Y	Y	Y	Y	Y
R^2	0.25	0.16	0.05	0.08	0.22	0.21	0.08	0.07
Obs.	290				286			

Note: Non-Financial Corporate Sector, 1947Q1-2021Q3. In the left (right) panel, we first calculate a 2-year (3-year) moving average for each variable from the original quarterly series, and then calculate the quarter-to-quarter growth rate of these moving average terms. *** : $p < 1\%$. The numbers in the brackets are the standard errors.

Table A.4: Regression coefficients, US time series

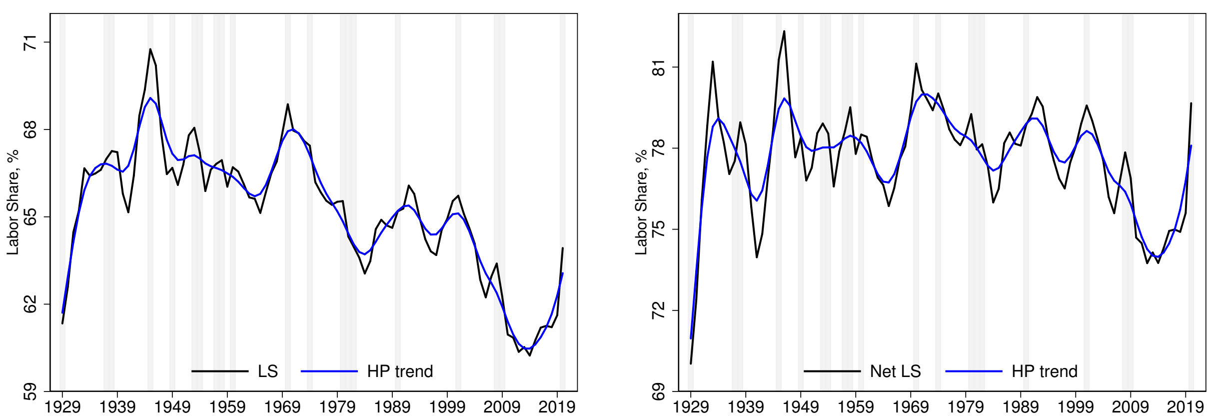
	Dep. Var.: ΔLS, 4-Year MA				Dep. Var.: ΔLS, HP trend			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
ΔLP	-0.38^{***}				-0.25^{***}			
	(0.06)				(0.05)			
$\Delta Wage$		0.46^{***}				0.37^{***}		
		(0.05)				(0.04)		
ΔEmp			0.18^{***}				0.25^{***}	
			(0.04)				(0.03)	
$\Delta Vadd$				-0.03				0.07^{*}
				(0.04)				(0.04)
Recession FE	Y	Y	Y	Y	Y	Y	Y	Y
Linear time trend	Y	Y	Y	Y	Y	Y	Y	Y
R^2	0.15	0.27	0.11	0.03	0.09	0.24	0.16	0.02
Obs.	282				298			

Note: Non-Financial Corporate Sector, 1947Q1-2021Q3. In the left (right) panel, we first calculate a 4-year moving average (HP trend with a smoothing parameter 1600) for each variable from the original quarterly series, and then calculate the quarter-to-quarter growth rate of these moving average (HP trend) terms. *** : $p < 1\%$. The numbers in the brackets are the standard errors.

Table A.5: Sample Correlation, Non-farm Business sector

Variable	LP	WAGE	EMP	VADD
Quarterly	-0.63^{***}	0.48^{***}	-0.18^{***}	-0.54^{***}
9-quarter MA	-0.44^{***}	0.33^{***}	0.21^{***}	-0.13^{***}
13-quarter MA	-0.39^{***}	0.36^{***}	0.22^{***}	-0.12^{**}
17-quarter MA	-0.31^{***}	0.40^{***}	0.29^{***}	-0.02
HP trend	-0.30^{***}	0.36^{***}	0.21^{***}	-0.07

Note: This table gives the correlation between growth rate in LS and in other labor market variables at various frequencies. $^{***}: p < 1\%$; $^{**}: p < 5\%$; $^{*}: p < 10\%$. Data is for the non-farm business sector in the US, from 1947Q1-2021Q4. Data source for LS is U.S. Bureau of Labor Statistics (BLS), Non-farm Business Sector: Labor Share [PRS85006173], retrieved from FRED, Federal Reserve Bank of St. Louis; The BLS LS assumes wage for proprietors is the same as the rest of economy. The results are qualitatively the same if we calculate LS using the definition in the main text.



(a) Gross Labor Share

(b) Net Labor Share

Figure A.1: Gross and Net Labor Share, 1929-2020

Note: This figure plots the gross and net labor income share in GDP, at the annual frequency, from 1929 to 2020. Also plotted are the HP trend with a smoothing parameter of 6.25 and the NBER dating of recessions. A year is labeled recession if at least two quarters of that year are in recession according to NBER recession dating.

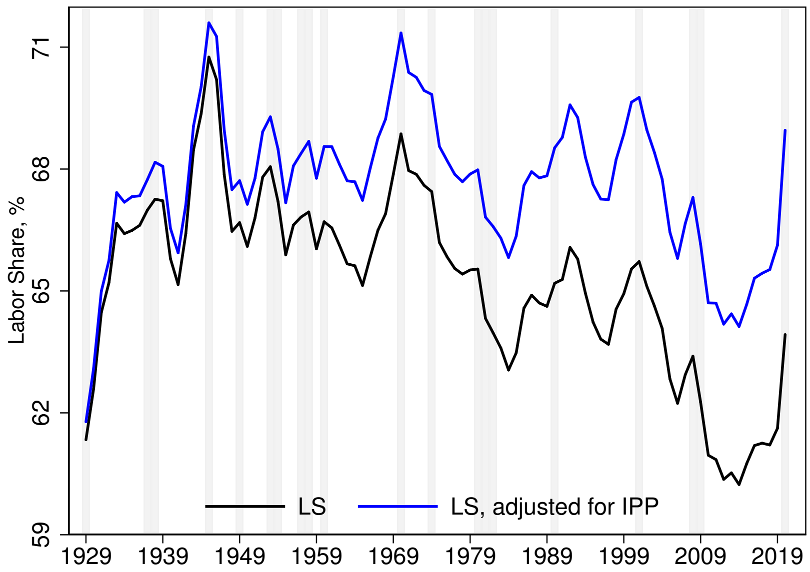


Figure A.2: LS w/ and w/o IPP adjustment, 1929-2020

Note: The gross labor share (LS) is calculated based on the formula provided in Section 2. IPP stands for intellectual property products. ”LS, adjusted for IPP” is calculated following Koh et al. (2021). In particular, we use GDP minus the value of IPP investment in each year as the denominator in calculating the adjusted labor share. A year is labeled recession if at least two quarters of that year are in recession according to NBER recession dating.

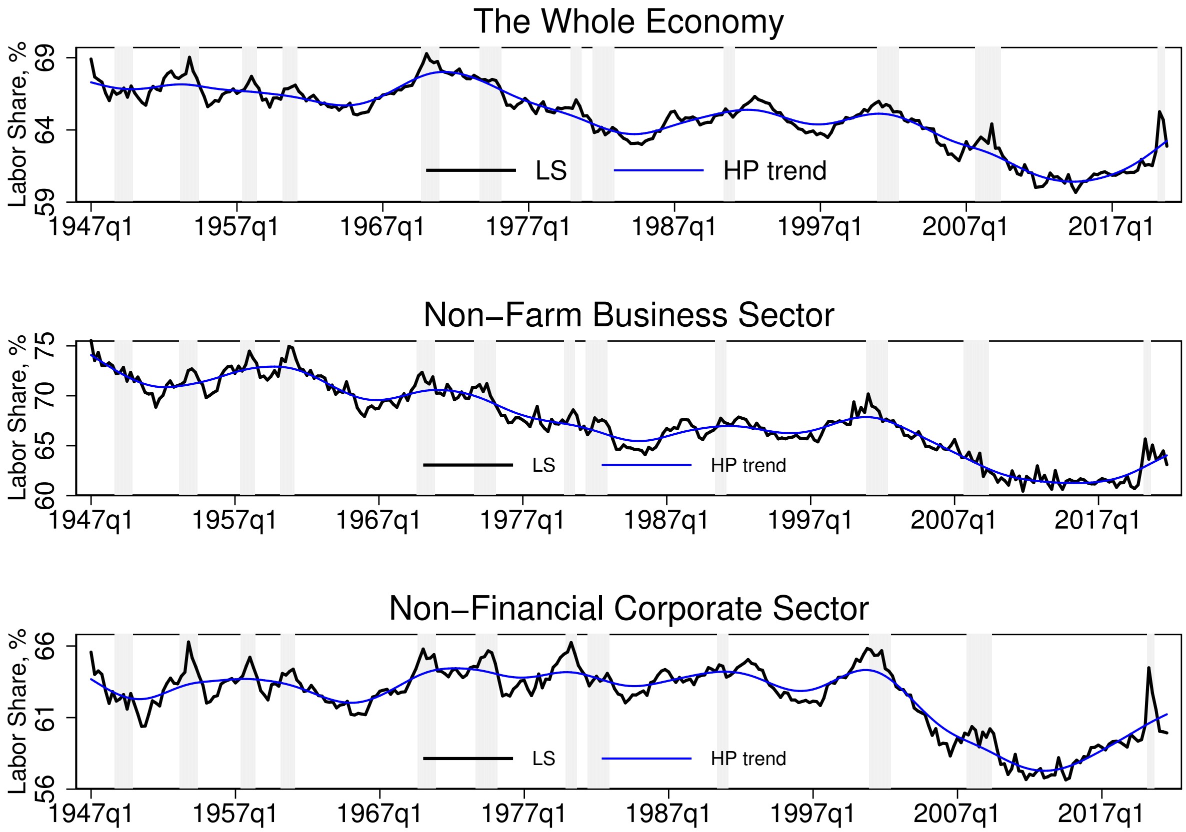
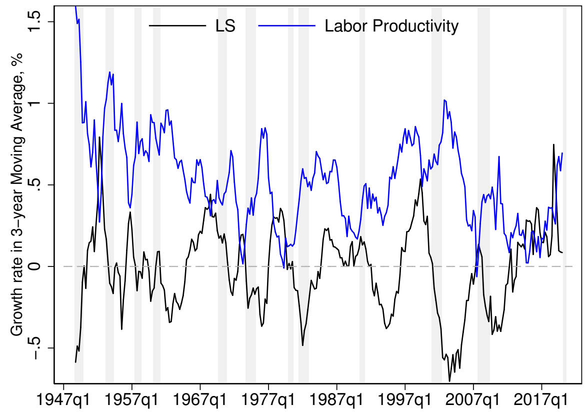
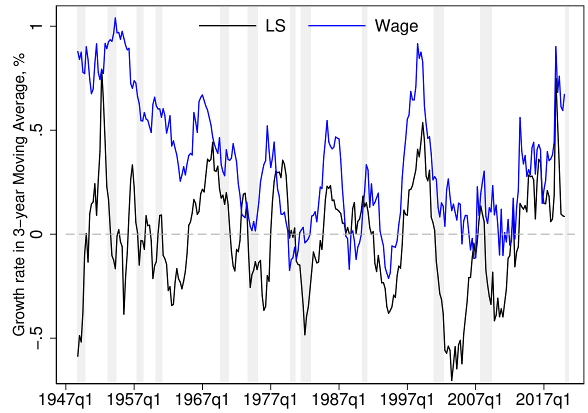


Figure A.3: LS in the Whole Economy, in the Non-Farm Business Sector, and in the Non-Financial Corporate Sector

Note: The smoothing parameter in HP filter is 1600. The method to calculate the LS in the whole economy and in the Nonfarm Business sector is detailed in Section 2. Data source for LS in the Non-Financial Corporate sector, which does not involve the issue of self employment, is U.S. Bureau of Labor Statistics, Non-financial Corporations Sector: Labor Share [PRS88003173], retrieved from FRED, Federal Reserve Bank of St. Louis;



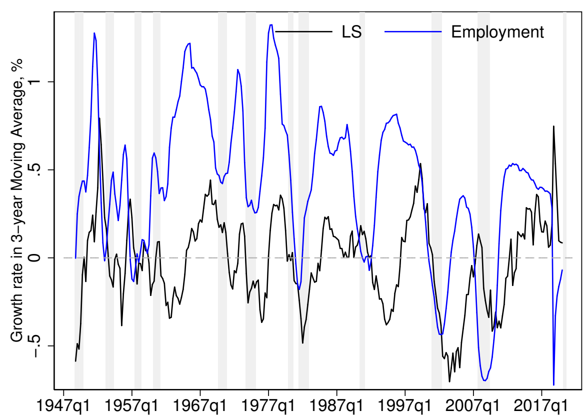
(a) LS and Labor productivity



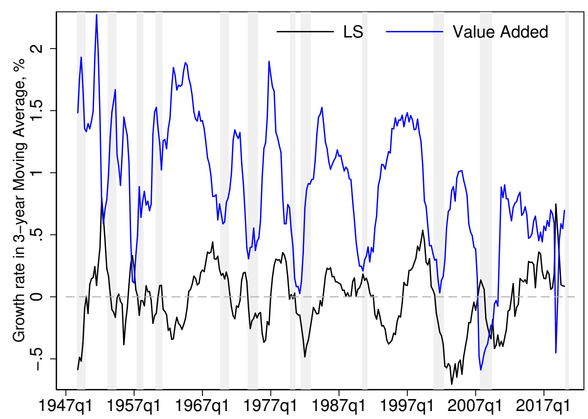
(b) LS and Wage

Figure A.4: Growth Rate in LS and Labor Productivity & Wage, 3-Year Moving Average

Note: To produce this figure we first calculate the 3-year moving average of LS, labor productivity and wage for the non-financial corporate sector, based on the original quarterly series. We then calculate, at each quarter, the growth rate from the previous quarter for the moving average terms.



(a) LS and Employment



(b) LS and Real Value Added

Figure A.5: Growth Rate in LS and Employment & Real Value Added, 3-Year Moving Average

Note: See Figure A.4.

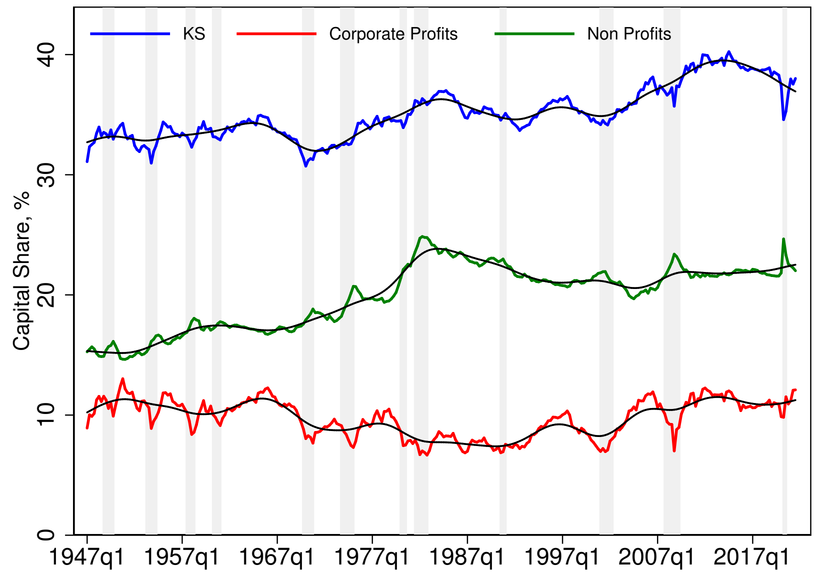


Figure A.6: Capital Share and Its Components

Note: This figure plots the share of capital income and of its components, at the quarterly frequency. Also plotted are the HP trends with a smoothing parameter of 1600 and the NBER dating of recessions.

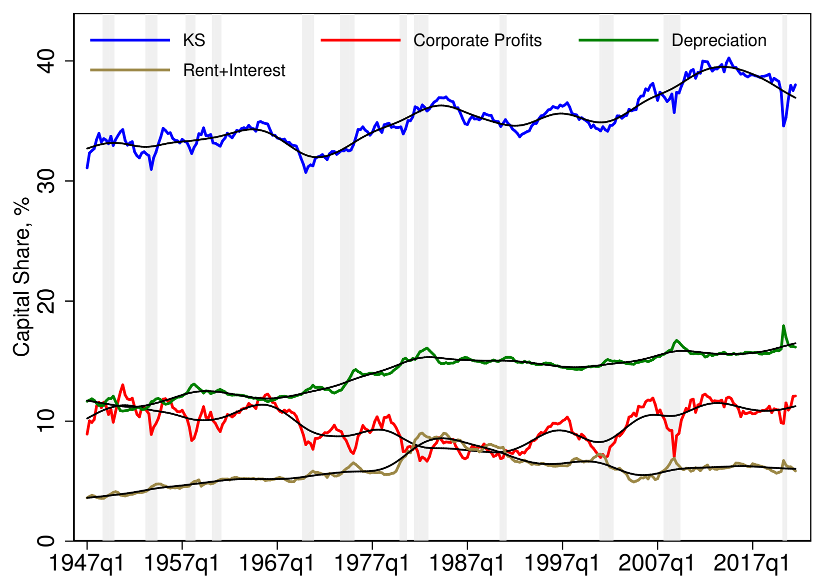


Figure A.7: Capital Share and Its Components

Note: This figure plots the share of capital income and of its components, at the quarterly frequency. Also plotted are the HP trends with a smoothing parameter of 1600 and the NBER dating of recessions.

Appendix B International Evidence

For countries outside of the United States, we use the STructural ANalysis (STAN) database for Australia, Canada, Denmark, Finland, France, Italy, Japan, Norway, Spain, Sweden and UK during the period 1970-2018.[31] We use the Business sector aggregates, excluding Households, Nonprofit Institutions, and the General Government. LS is defined as the ratio of *Compensation of Employees* to *Value added*.[32] Figure B.1 presents LS in the 11 selected OECD countries.

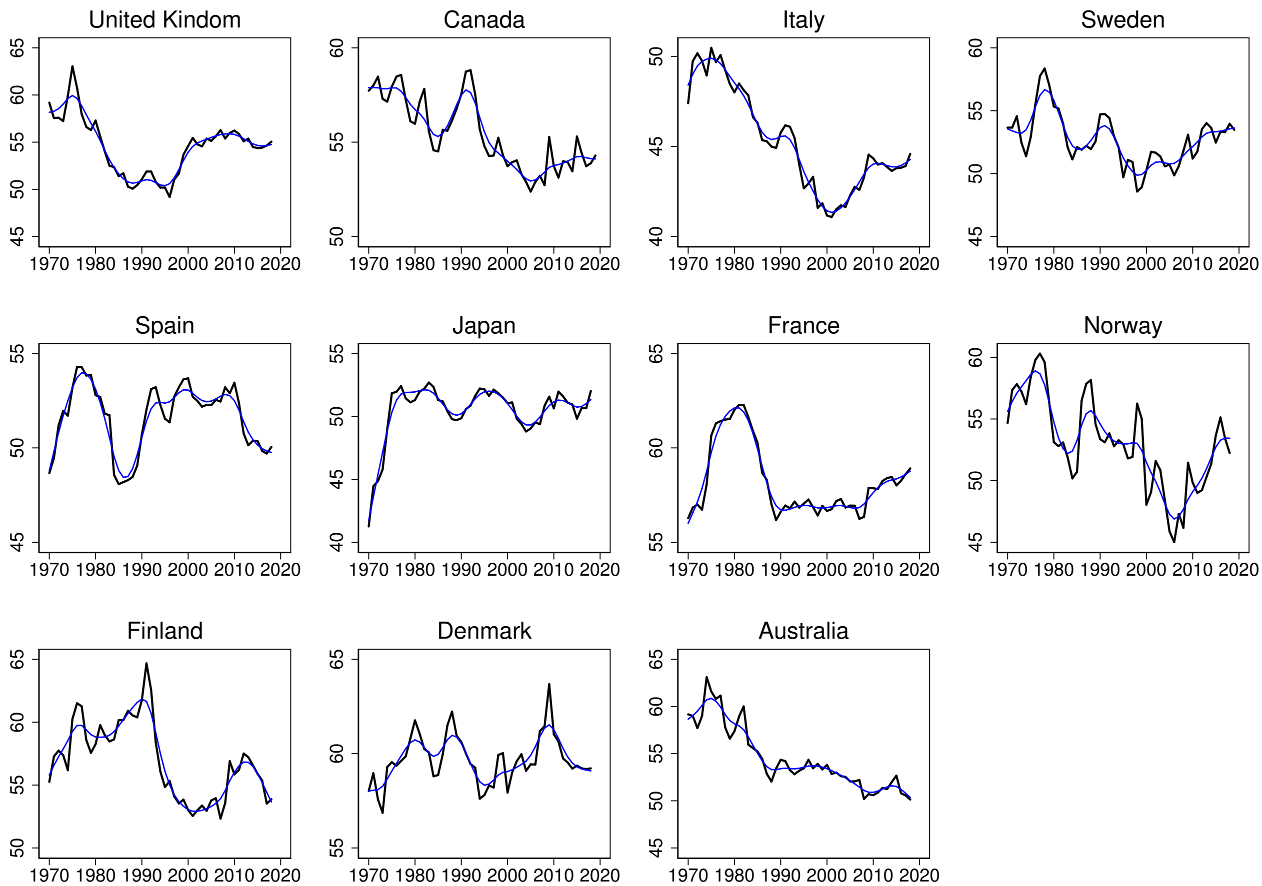


Figure B.1: LS in the Business Sector (%, in black) and the HP trend (in blue) in Selected OECD Countries

Note: The smoothing parameter used in obtaining the HP trend of the annual LS series is 6.25. *Data Source:* The STructural ANalysis (STAN) Database from OECD.Stat.

[31] 1970 is the starting year for 9 out of 11 countries; for Australia and Spain it is 1974 and 1980, respectively. We do not include Germany, because the unification makes its data available only since 1991.

[32] For Canada, Italy, Spain, and Sweden, data on 'Tax less subsides on production' for the business sector is only available after 1995 or later. The STAN database does not contain proprietor's income and the series for self employment are available only after 1995, hence no adjustment is made.

We treat the data in the same way as for the US.[33] The left panel of Table B.2 reports the regression result for 3-Year moving averages. The correlation is similar to the one we estimated with the US data. In particular, there is a negative correlation between the growth rate of the LS and those of labor productivity and value added, and there is a positive correlation between the growth rate of the LS and those of wages and employment.[34] In Table B.1 and the right panel of Table B.2, we show the results for the original annual time series data and the HP trend. At the annual frequency, the correlation between change in LS and in employment is positive. For the HP trend, while correlations for all other variables are the same as 3-year moving averages, the correlation between change in LS and in value added becomes insignificant.

Table B.1: Regression Coefficients, OECD panel data

	Dep. var.: ΔLS, annual			
	(1)	(2)	(3)	(4)
ΔLP	-0.37^{***}			
	(0.03)			
$\Delta Wage$		0.43^{***}		
		(0.02)		
ΔEmp			0.07^{*}	
			(0.04)	
$\Delta Vadd$				-0.18^{***}
				(0.07)
Country FE	Y	Y	Y	Y
Year	Y	Y	Y	Y
R^2	0.23	0.55	0.01	0.10
Obs.	516	516	529	516

Note: Data is for the Business sector for 11 OECD countries from 1970-2018.

[33]In STAN, hours data is only available since 1995 or later for most countries. We define labor productivity as real value added divided by employment and deflate nominal compensation by the value added deflator, the real wage rate is real total compensation divided by employment.

[34]The results for 2- and 4-year moving averages are similar, both in sign and significance, to Table B.2. The only difference is that the coefficient for value added, at the 4-year moving average, is negative but insignificant.

Table B.2: Regression Coefficients, 3-Year Moving Average

	Dep. var.: ΔLS, 3-year MA				Dep. var.: ΔLS, HP trend			
	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
ΔLP	-0.30^{***}				-0.13^{***}			
	(0.03)				(0.03)			
$\Delta Wage$		0.40^{***}				0.33^{***}		
		(0.02)				(0.01)		
ΔEmp			0.11^{***}				0.11^{***}	
			(0.03)				(0.02)	
$\Delta Vadd$				-0.06^{***}				0.035
				(0.02)				(0.021)
Country FE	Y	Y	Y	Y	Y	Y	Y	Y
Year	Y	Y	Y	Y	Y	Y	Y	Y
R^2	0.18	0.53	0.05	0.04	0.08	0.54	0.08	0.05
Obs.	494	494	507	494	516	516	529	516

Note: Data is for the Business sector for 11 OECD countries from 1970-2018. We first calculate the 3-year moving average and the HP trend (with a smoothing parameter of 6.25) for each variable from the original annual series, and then calculate the year-to-year growth rate of these moving average and HP trend terms.

Appendix C Derivations and Proofs - For Online Publication

C.1 Proofs of Propositions in the Baseline Model

Proof of proposition 1 We show how to derive the price formula for the wage and interest rates and prove why there are at most two consecutive qualities of capital employed in producing the consumption good. Recall that $q_j(t)$ is the price of capital j in units of time-t marginal utility, hence $p_j(t) = q_j(t)c(t)$, is the same price in units of time-t consumption and $e^{-\rho(s-t)}\frac{c(t)}{c(s)}$ is the relative price of time-s consumption to time-t consumption. The following non-arbitrage condition holds,

$$p_j(t) = q_j(t)c(t) = r_j(t)\Delta + e^{-\rho\Delta}\frac{c(t)}{c(t+\Delta)} * q_j(t+\Delta)c(t+\Delta), \quad \text{as } \Delta \to 0$$

It follows that

$$\begin{aligned}
r_j(t) &= -\frac{1}{\Delta}\left\{e^{-\rho\Delta}[q_j(t)+\dot{q}_j(t)\Delta]c(t) - q_j(t)c(t)\right\} \\
&= -\dot{q}_j(t)c(t) + \rho q_j(t)c(t), \quad \text{as } \Delta \to 0 \\
&= q_j(t)c(t)[-\frac{\dot{q}_j(t)}{q_j(t)} + \rho] \\
&= q_j(t)c(t)b
\end{aligned}$$

The last equality follows as we know that the price of capital, in units of time-t marginal utility, decreases at the rate of $b-\rho$. Therefore

$$q_j(t) = \frac{1}{bc(t)} r_j(t) = \frac{1}{bc(t)} \frac{\gamma^{j'-j}-1}{\gamma^{j-j'}-1/a^{j-j'}}.$$

To see why there are at most two vintages of capital simultaneously used in production and they must be of adjacent vintages, consider the case where capital j' is used. The price of capital $j'+1$ is $q_{j'+1}(t) = \frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(t)}$. Zero profit in the innovation sector implies that for any capital $j > j'+1$, its price equals $a^{j-j'-1}q_{j'+1}(t) = a^{j-j'-1}\frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(t)}$, which is strictly larger than the value it would obtain in the production of consumption, $\frac{\gamma^{j-j'}-1}{\gamma^{j-j'}-1/a^{j-j'}}\frac{1}{bc(t)}$. Therefore, any capital of vintage larger than $j'+1$ will not be employed in production of consumption.

Proof of proposition 2 Consider an adoption phase when capital j and $j+1$ are both used to produce consumption. Capital $j+1$ self accumulates during this phase and its price $q_{j+1}(t)$ decreases at the rate $b-\rho$, which implies that consumption grows at the rate of $b-\rho$. Labor is fully employed by capital j at the beginning of the adoption phase, and by capital $j+1$ at the end of it. Output therefore increases from γ^j at the beginning to γ^{j+1} at the end. As the rate of increase in consumption is $b-\rho$, the length of the growth phase is $\frac{\log\gamma}{b-\rho}$.

At the end of the adoption phase, capital $j+1$ employs all the labor force and, according to Proposition 1, the price of capital $j+1$ is, $q_{j+1}(t) = \frac{\gamma-1}{\gamma/a}\frac{1}{bc(t)}$. Zero profits in the innovation sector imply that the price of capital $j+2$ satisfies $q_{j+2}(t) = aq_{j+1}(t) = a\frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(t)}$. However, the value of employing capital $j+2$ in producing consumption, in this moment, is $v_{j+2}(t) = \frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(t)}$. As

$$q_{j+2}(t) = a\frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(t)} > v_{j+2}(t) = \frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(t)},$$

using capital $j+2$ to produce additional consumption at the end of the adoption phase for capital $j+1$ would yield negative profits. Hence capital $j+1$ will accumulate further, which decreases the prices of both vintages, $j+1$ and $j+2$. The left hand side (LHS) of the last inequality decreases at the rate of $b-\rho$ while its right hand side (RHS) remains constant. Capital $j+2$ will be used to produce consumption when the LHS equals RHS.[35] As the price of capital decreases at the rate of $b-\rho$, the innovation phase lasts for $\frac{\log a}{b-\rho}$ units of time. Note that consumption remains stagnant during the innovation phase, as all labor is already employed by capital $j+1$.

Proof of Proposition 3 We first investigate the evolution of the stock of capital. Start by calculating how much capital is transformed into that of a more advanced vintage, at the beginning of an adoption phase, once the economy enters the recurring cycles. Consider an adoption phase when capital j and $j+1$ are simultaneously employed in production. In the main text, we denote with τ_{j+1} the beginning of such a phase. Here for simplicity we normalize τ_{j+1} to 0. At time 0, $a * k_j^3(0)$ units of capital j are converted into $k_{j+1}(0)$ units of capital $j+1$ by using the innovation technology. The remaining stock, $k_j^1(0) = \gamma^j$ is used to produce consumption $c(0) = \gamma^j$.

Denote with $\sigma_j(t)$ the fraction of labor employed by capital j during an adoption phase. Total output of the consumption good can then be written as

$$\gamma^j \sigma_j(t) + \gamma^{j+1}(1 - \sigma_j(t)) = c(t)$$

It follows that $\sigma_j(t) = \frac{\gamma^{j+1} - c(t)}{\gamma^{j+1} - \gamma^j} = \frac{\gamma^{j+1} - \gamma^j e^{(b-\rho)t}}{\gamma^{j+1} - \gamma^j} = \frac{\gamma - e^{(b-\rho)t}}{\gamma - 1}$, where the second equality holds because, in the adoption phase, consumption increases at the rate of $b-\rho$. Note that when $t = \frac{\log \gamma}{b-\rho}$, $\sigma_j(t) = 0$. Thus, at the end of the adoption phase, all labor is employed by capital $j+1$.

Assume that capital j is converted into capital $j+1$ as soon as it is freed from use in producing the consumption good during the adoption phase.[36] That is,

$$\begin{aligned} k_j^3(t) &= -dk_j(t) = -\gamma^j * d\sigma_j(t) \\ &= \gamma^j \frac{b-\rho}{\gamma - 1} e^{(b-\rho)t} * dt \end{aligned}$$

[35] A different, more technical, interpretation of this (in-)equality is the following. The overall optimal control problem can be divided into a series of sub-problems, each dealing with the period of time during which two consecutive capital goods are used. Denote $\lambda_j(J)$ and $\lambda_{j+1}(J)$ the co-state variables for capital j and $j+1$, respectively, in the sub-problem J. A necessary condition for equivalence of the original problem and the series of sub-problems is that $\lambda_{j+1}^{end}(J) = \lambda_{j+1}^0(J+1)$. That is, the multiplier for capital $j+1$ at the end of sub-problem J should equal that at the beginning of sub-problem $J+1$. This is the price condition reported here.

[36] Alternatively, we can assume that capital j released from production is first self-accumulated from time t for a period of positive length Δ_t, and converted to capital $j+1$ altogether at $t+\Delta_t$. These two assumptions are equivalent in the sense that they deliver exactly the same amount of capital $j+1$ at time $t+\Delta_t$.

Therefore, during the adoption phase, the law of motion for capital $j+1$ is[37]

$$\begin{aligned} dk_{j+1}(t) &= bk_{j+1}^2(t)dt - k_{j+1}^3(t) + \frac{k_j^3(t)}{a} \\ &= b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))]dt - 0 + \frac{1}{a} * \gamma^j \frac{b-\rho}{\gamma-1} e^{(b-\rho)t} dt \end{aligned}$$

Equivalently,

$$\begin{aligned} \dot{k}_{j+1}(t) &= b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))] + \gamma^j \frac{b-\rho}{a(\gamma-1)} e^{(b-\rho)t} \\ &= b[k_{j+1}(t) + \frac{\gamma^{j+1}}{\gamma-1}] + \gamma^j \frac{b(1-a\gamma)-\rho}{a(\gamma-1)} e^{(b-\rho)t} \end{aligned}$$

until $t = \tau^g \equiv \frac{\log\gamma}{b-\rho}$ when the adoption phase ends. The solution to this ordinary differential equation has the following form: $k_{j+1}(t) = \theta_0 + \theta_1 e^{bt} + \theta_2 e^{(b-\rho)t}$. Differentiating both sides w.r.t. time t and matching coefficients in common terms, we have

$$\theta_0 = -\frac{\gamma^{j+1}}{\gamma-1},$$

and

$$\theta_2 = \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)}.$$

Substituting these back into the formula for $k_{j+1}(t)$,

$$k_{j+1}(t) = -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 e^{bt} + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)} e^{(b-\rho)t}.$$

Using the initial condition at time $t=0$,

$$k_{j+1}(0) = -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)},$$

we have,

$$\theta_1 = k_{j+1}(0) - \gamma^j \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}.$$

[37]Note that here we assume that before capital $j+2$ is used in producing consumption goods, say at $t=\tau_{j+2}$, capital $j+1$ will only be used in (producing the consumption goods and) replicating itself, and not be used in creating capital $j+2$. Alternatively, we can assume that any capital $j+1$ beyond the necessary amount in producing consumption goods is converted immediately to capital $j+2$. The amount of capital $j+2$ obtained at $t=\tau_{j+2}$ under the two assumptions would be the same. In addition, as capital $j+2$ will not be used in producing consumption goods before $t=\tau_{j+2}$, the price of capital $j+1$ is determined as before, and the (implied) price of capital $j+2$ is also not altered.

At time $t = \tau^g \equiv \frac{\log \gamma}{b-\rho}$,

$$
\begin{aligned}
k_{j+1}(\tau^g) &= -\frac{\gamma^{j+1}}{\gamma - 1} + \theta_1 \gamma^{\frac{b}{b-\rho}} + \gamma^j \frac{(a\gamma - 1)b + \rho}{\rho a(\gamma - 1)} \gamma \\
&= \gamma^{j+1} + (k_{j+1}(0) - \gamma^j \tilde{x}) \gamma^{\frac{b}{b-\rho}} + \gamma^{j+1} \tilde{x}
\end{aligned}
$$

where $\tilde{x} \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$.

The innovation phase comes next and lasts until $t = \frac{\log \gamma + \log a}{b-\rho}$. During the innovation phase, $k_j^3(t) = 0$ as capital j has been used up; and $k_{j+1}^1 = \gamma^{j+1}$ as labor is all and only employed by capital $j+1$. Assume $k_{j+1}^3(t) = 0$.[38] The dynamics for $k_{j+1}(t)$ is

$$dk_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}]dt.$$

Solve this differential equation, and capital $j+1$ satisfies

$$k_{j+1}(t) = \gamma^{j+1} + e^{b(t-\tau^g)}[k_{j+1}(\tau^g) - \gamma^{j+1}], \quad \text{for } \tau^g \leq t \leq \tau^g + \tau^n,$$

where $k_{j+1}(\tau^g)$ is the amount of capital $j+1$ at $t = \tau^g$. When $t = \tau^g + \tau^n = \frac{\log \gamma + \log a}{b-\rho}$, capital $j+1$ is

$$k_{j+1}(\tau^g + \tau^n) = \gamma^{j+1} + a^{\frac{b}{b-\rho}}[k_{j+1}(\tau^g) - \gamma^{j+1}].$$

At time $t = \tau^g + \tau^n$, γ^{j+1} units of capital are employed in producing the consumption good, and the remaining capital of vintage $j+1$, $k_{j+1}(\tau^g + \tau^n) - \gamma^{j+1}$, is converted to capital $j+2$. It follows that,

$$
\begin{aligned}
k_{j+2}(\tau^g + \tau^n) &= \frac{1}{a}[k_{j+1}(\tau^g + \tau^n) - \gamma^{j+1}] \\
&= \frac{a^{\frac{b}{b-\rho}}}{a}[k_{j+1}(\tau^g) - \gamma^{j+1}] \\
&= a^{\frac{\rho}{b-\rho}}\{[k_{j+1}(0) - \gamma^j \tilde{x}]\gamma^{\frac{b}{b-\rho}} + \gamma^{j+1}\tilde{x}\}
\end{aligned}
$$

[38] Note that here we made the assumption that, before $t = \frac{\log \gamma + \log a}{b-\rho}$, capital $j+1$ is only used in replicating itself and not used in creating $j+2$. Both activities satisfy the zero profit conditions. In essence, between $t = \frac{\log \gamma}{b-\rho}$ and $t = \frac{\log \gamma + \log a}{b-\rho}$, various arrangements regarding what percentage of and when non-production capital $j+1$ is converted into capital $j+2$ are equivalent as they generate the same amount of capital $j+2$ at $t = \frac{\log \gamma + \log a}{b-\rho}$.

again $\tilde{x} \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$. Equivalently,

$$\begin{aligned}
\frac{k_{j+2}(\tau^g+\tau^n)}{\gamma^{j+1}} &= a^{\frac{\rho}{b-\rho}}\left\{[\frac{k_{j+1}(0)}{\gamma^j}-\tilde{x}]\gamma^{\frac{\rho}{b-\rho}}+\tilde{x}\right\}, \\
&= (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}}-1)\tilde{x}, \\
&\equiv (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - x.
\end{aligned}$$

with $x \equiv a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}}-1)\tilde{x} = a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}}-1)\frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$. This is the formula we showed in the text. From this equation one concludes that there exits a unique steady state value of the normalized capital stock, $k^* \equiv \frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$, with τ_{j+1} the first time capital $j+1$ used in the production of consumption, which satisfies,

$$k^* = \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}\frac{(a\gamma)^{\frac{\rho}{b-\rho}}-a^{\frac{\rho}{b-\rho}}}{(a\gamma)^{\frac{\rho}{b-\rho}}-1}.$$

Denote by $j=0$ the least advanced capital vintage. The economy enters a recurring cycle when capital 1 is created and employed in production, call τ_1 this moment. That $k_1(\tau_1)=k^*$ follows from the Transversality condition. Note that each vintage of capital has a finite 'life cycle' in our model, it is therefore sufficient to show that the Transversality condition holds at a certain point in the 'life cycle' for each vintage of capital. Without loss of generality, we choose the time when a vintage of capital is firstly introduced, i.e. τ_{j+1} for capital $j+1$. The Transversality condition reads,

$$0 = \lim_{j\to\infty} e^{-\rho\tau_{j+1}}\frac{k_{j+1}(\tau_{j+1})}{c(\tau_{j+1})} = \lim_{j\to\infty} e^{-\rho\tau_1}(a\gamma)^{\frac{-\rho}{b-\rho}j} * \frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$$

where the last equality follows from the fact that $\tau_{j+1} = j * \frac{\log a+\log\gamma}{b-\rho}+\tau_1$ and $c(\tau_{j+1})=\gamma^j$. From the law of motion for $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$ above, we have that $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j}-k^* = (a\gamma)^{\frac{\rho}{b-\rho}}(\frac{k_j(\tau_j)}{\gamma^{j-1}}-k^*) = (a\gamma)^{\frac{\rho}{b-\rho}j}(k_1(\tau_1)-k^*)$. That is, $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j} = k^*+(a\gamma)^{\frac{\rho}{b-\rho}j}(k_1(\tau_1)-k^*)$. Substitute this formula into the Transversality condition,

$$\begin{aligned}
0 &= \lim_{j\to\infty} e^{-\rho\tau_1}(a\gamma)^{\frac{-\rho}{b-\rho}j} * [k^*+(a\gamma)^{\frac{\rho}{b-\rho}j}(k_1(\tau_1)-k^*)], \\
&= \lim_{j\to\infty} e^{-\rho\tau_1}[k_1(\tau_1)-k^*].
\end{aligned}$$

It follows that $k_1(\tau_1) = k^*$. The value of $k_1(\tau_1)$ is endogenously determined in the initial cycle, to which we turn next.

Denote with $k_0(0)$ the initial value of capital 0. Start with the case $0 < k_0(0) < 1$. That is, there is not enough initial capital to employ all labor force. We need to determine how to allocate the initial capital, at $t = 0$, between producing consumption and self-accumulation. Denote with $k_0^1(0)$ the units of capital allocated to produce consumption.

The price of capital in terms of current marginal utility of consumption is $q_0(t) = \frac{1}{bc(t)}$.[39] Thus, consumption grows at the rate of $b - \rho$. The dynamics of $k_0(t)$ is

$$\begin{aligned}\dot{k}_0(t) &= b[k_0(t) - k_0^1(t)] \\ &= b[k_0(t) - k_0^1(0)e^{(b-\rho)t}]\end{aligned}$$

The solution to this ordinary differential equation is of the form: $k_0(t) = \phi_0 + \phi_1 e^{bt} + \phi_2 e^{(b-\rho)t}$. Differentiating this expression and matching coefficients with the formula above gives

$$\phi_0 = 0, \quad \phi_2 = k_0^1(0)\frac{\rho}{b}$$

Further, use the initial condition to obtain $\phi_1 = k_0(0) - \frac{b}{\rho}k_0^1(0)$. This initial adoption phase ends when $c(\tau_0^g) = k_0^1(0)e^{(b-\rho)\tau_0^g} = 1$, that is, at $\tau_0^g = \frac{1}{b-\rho}\log\frac{1}{k_0^1(0)}$. The capital stock at $t = \tau_0^g$ is

$$k_0(\tau_0^g) = k_0(0) * k_0^1(0)^{\frac{-b}{b-\rho}} - \frac{b}{\rho}k_0^1(0)^{\frac{-\rho}{b-\rho}} + \frac{b}{\rho}.$$

The economy then enters its first innovation phase where the dynamics of capital is given by

$$\dot{k_0}(t) = b[k_0(t) - 1].$$

The solution to this differential equation is

$$k_0(t) = 1 + e^{b(t-\tau_0^g)}[k_0(\tau_0^g) - 1]$$

To determine the length of the first innovation phase, note that the price of capital of type $j = 0$ at $t = \tau_0^g$ is $q_0(\tau_0^g) = \frac{1}{bc(\tau_0^g)} = \frac{1}{b}$. From the zero profit condition for innovation, the (implicit) price of capital of type $j = 1$ is now $q_1^*(\tau_0^g) = aq_0(\tau_0^g) = \frac{a}{b}$. Denote with τ_1 the first time when capital 1

[39] This is obtained from the Euler equation. The rental price of capital and the wage, in units of the current consumption good, are 1 and 0, respectively.

is created and employed in production. The length of this innovation phase is therefore $\tau_1 - \tau_0^g$. The price of capital of type 1 at $t = \tau_1$ is $q_1(\tau_1) = \frac{\gamma-1}{\gamma-1/a}\frac{1}{b}$. As the price of capital decreases at the rate of $b - \rho$ during the innovation phase, we have

$$q_1^*(\tau_0^g)e^{-(b-\rho)(\tau_1-\tau_0^g)} = q_1(\tau_1)$$

The length of the phase is

$$\tau_1 - \tau_0^g = \frac{1}{b-\rho}\log(\frac{a\gamma-1}{\gamma-1})$$

Note that this is different from the length of an innovation phase after the economy enters the recurring cycles. At $t = \tau_1$, the value of capital 0 is

$$k_0(\tau_1) = 1 + (\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}[k_0(\tau_0^g) - 1],$$

of which 1 unit is used in producing the consumption good, and the remaining $k_0(t_1) - 1$ units converted into capital of type 1. Therefore the amount of capital 1 at $t = \tau_1$ is

$$\begin{aligned}
k_1(\tau_1) &= \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}[k_0(\tau_0^g) - 1] \\
&= \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}[k_0(0) * k_0^1(0)^{\frac{-b}{b-\rho}} - \frac{b}{\rho}k_0^1(0)^{\frac{-\rho}{b-\rho}} + \frac{b}{\rho} - 1] \\
&= \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}\left\{k_0(0)^{\frac{-\rho}{b-\rho}}\chi^{\frac{-b}{b-\rho}}[1 - \frac{b}{\rho}\chi] + \frac{b}{\rho} - 1\right\}
\end{aligned}$$

where $\chi \equiv \frac{k_0^1(0)}{k_0(0)}$ is the fraction of capital 0 used in producing the consumption good. The Transversality condition we derived before requires that $k_1(\tau_1) = k^*$. Note $k_1(\tau_1)$ is a strictly decreasing function of χ. As $\chi \to 0$, $k_1(\tau_1) \to \infty$. On the other hand, as $\chi \to 1$, $k_1(\tau_1) \to \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}(\frac{b}{\rho} - 1)(1 - k_0(0)^{\frac{-\rho}{b-\rho}}) < 0$. The monotonicity of $k_1(\tau_1)$ guarantees existence and uniqueness of a $k_0^1(0)$ that satisfies the Transversality condition.

C.2 Factor Shares for the Whole Economy

In the main text we focused on the factor shares of the consumption sector instead of those for the whole economy as our simplifying assumptions set the share of labor equal to zero in the other two sectors. Here we compute the labor share for the whole economy under these simplifying assumptions and derive conditions under which the main results carry through. Consider an adoption phase where capital j and $j+1$ are simultaneously used in production of consumption. Denote with $p(t)$ the relative price of the investment good to the consumption good. This will be the new capital $j+1$, which is being accumulated and adopted while capital of type j is being phased out. Recall that the price of capital $j+1$ in units of current marginal utility is $q_{j+1}(t)$, hence $p(t) = q_{j+1}(t)c(t)$. Denote with τ_{j+1} the starting time of the adoption phase. At $t = \tau_{j+1}$, the gross LS in the whole economy is[40]

$$\tilde{LS} = \frac{w}{c + p\dot{k}_{j+1}} = \frac{w/c}{1 + \frac{\gamma-1}{\gamma-1/a}} \frac{\dot{k}_{j+1}}{bc}.$$

where $\frac{w}{c}$ is the LS in the consumption sector. As $\frac{\dot{k}_{j+1}}{c}(\tau_{j+1}$ is a constant, corresponding to the normalized steady state for all j, the aggregate LS is a constant multiple of the consumption sector's LS. Hence, the aggregate LS declines together with $\frac{w}{c}$ from $t = \tau_{j+1}$ until the end of the adoption phase, and the decline is independent of the capital vintage considered.

For $t > \tau_{j+1}$,

$$\begin{aligned}
\tilde{LS}(t) &= \frac{w(t)}{c(t) + p(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + q_{j+1}(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^j}{bc(t)}[b(\frac{k_{j+1}(\tau_{j+1})}{\gamma^j} - \bar{x})e^{b(t-\tau_{j+1})} + (b-\rho)\bar{x}e^{(b-\rho)(t-\tau_{j+1})}]}
\end{aligned}$$

where $\bar{x} \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$ is a constant. Recall that $c(t) = \gamma^j e^{(b-\rho)(t-\tau_{j+1})}$ during the adoption phase. Again $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$ is a constant in steady state. Therefore the denominator is a function of $t - \tau_{j+1}$, and independent of j itself. We have already shown that $w(t)/c(t)$, the labor share in the consumption sector, does not depend on j. Therefore, the aggregate labor share is also independent of capital vintages and follows the pattern of the LS of the consumption sector during the whole adoption phase.

During the innovation phase, the price of capital $j+1$, in terms of the consumption good, decreases at the rate of $b - \rho$ and consumption remains stagnant at γ^{j+1} while additional units

[40]In the model, the net value added is obtained by subtracting the value of depreciated capital in the innovation sector from gross value added. It is straightforward to show that the net LS displays the same cycles as the gross LS. Actually under the assumption that innovation occurs at the end of an innovation phase, the net and gross LS equals each other during the whole innovation phase but the last moment.

of capital of type $j+1$ are accumulated in the "background". The aggregate LS is

$$\begin{aligned}
\tilde{LS}(t) &= \frac{w(t)}{c(t)+p(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1+q_{j+1}(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1+\frac{\gamma-1}{\gamma-1/a}\frac{\gamma^j}{bc(t)}e^{-(b-\rho)(t-\tau^g_{j+1})}[be^{b(t-\tau^g_{j+1})}(\frac{k_{j+1}(\tau^g_{j+1})}{\gamma^j}-\gamma)]} \\
&= \frac{w(t)/c(t)}{1+\frac{\gamma-1}{\gamma-1/a}\frac{\gamma^{j+1}}{bc(t)}[be^{\rho(t-\tau^g_{j+1})}(\frac{k_{j+1}(\tau^g_{j+1})}{\gamma^j}-\gamma)]}
\end{aligned}$$

where τ^g_{j+1} denotes the ending (*resp.* beginning) time of the adoption (*resp.* innovation) phase. Same as in the adoption phase, both the numerator and denominator are independent of capital vintages and so is the aggregate LS.

It is straightforward to see that the aggregate LS still declines in the adoption phase: total output grows at an even faster rate than consumption during the adoption phase. During an innovation phase, the wage rate grows at the rate of $b-\rho$ to become, at the end of that phase, γ times larger than that at the beginning. The nominal value of investment (in units of current consumption good) grows at the rate ρ, and consumption remains constant. To retain the results that the LS increases during the innovation phase, we need that total output grows less than γ times during this phase, which is equivalent to,

$$\frac{1}{\gamma-1/a}[\frac{k_{j+1}(\tau^g_{j+1})}{\gamma^j}-\gamma](\gamma^{\frac{\rho}{b-\rho}}-\gamma)<1.$$

Using the formula for $k_{j+1}(\tau^g_{j+1})$ calculated before, this condition can be further simplified as

$$\frac{1}{1-1/(a\gamma)}\frac{\gamma^{\frac{\rho}{b-\rho}}-1}{(a\gamma)^{\frac{\rho}{b-\rho}}-1}(\gamma^{\frac{\rho}{b-\rho}}-\gamma)<1$$

which is satisfied if a is sufficiently greater than 1, and γ is relatively close to 1.

C.3 Proofs for the Extended Model with Endogenous Labor Supply

Consider again an adoption phase with capital j and $j+1$, normalize its beginning at $t=0$ and let $\sigma_j(t)$ be the fraction of labor employed by capital j. Total output of consumption is

$$\gamma^j l(t)\sigma_j(t) + \gamma^{j+1} l(t)(1-\sigma_j(t)) = c(t)$$

It follows that $\sigma_j(t) = \frac{\gamma^{j+1} - c(t)/l(t)}{\gamma^{j+1}-\gamma^j} = \frac{\gamma^{j+1} - \frac{c(0)}{l(0)} e^{\eta(b-\rho)t}}{\gamma^{j+1}-\gamma^j} = \frac{\gamma - e^{\eta(b-\rho)t}}{\gamma-1}$, where the second equality holds as consumption in the adoption phase increases at the rate of $b-\rho$, and employment decreases at the rate $(\eta-1)(b-\rho)$. This implies that at $t = \frac{\log\gamma}{\eta(b-\rho)}$ we have $\sigma_j(t) = 0$ and the adoption phase ends. Hence, with endogenous labor supply the length of the adoption phase shrinks from $\frac{\log\gamma}{b-\rho}$ to $\frac{\log\gamma}{\eta(b-\rho)}$.

At $t=0$, the following three conditions hold

$$c(0) = \gamma^j l(0); \quad \frac{w(0)}{c(0)} = \zeta l(0)^{\frac{1}{\eta-1}}; \quad w(0) = \gamma^j \frac{a-1}{a-1/\gamma}.$$

Therefore, we have[41]

$$l(0) = [\frac{a-1}{a-1/\gamma}\frac{1}{\zeta}]^{\frac{\eta-1}{\eta}}, \qquad c(0) = \gamma^j l(0).$$

At $t=0$, $a * k_j(0)$ units of capital j are converted into $k_{j+1}(0)$ units of capital $j+1$. The remaining amount, $k_j^1(t_0) = \gamma^j l(0)$, of capital is used to produce consumption $c(0) = \gamma^j l(0)$. At $t = \tau^g$, the labor supply, $l(\tau^g)$, is

$$l(\tau^g) = l(0) * e^{-(\eta-1)(b-\rho)\frac{\log\gamma}{\eta(b-\rho)}} = l(0)\gamma^{-\frac{\eta-1}{\eta}}.$$

The price of capital $j+1$ in units of current marginal utility, $q_{j+1}(\tau_g)$, is

$$q_{j+1}(\tau^g) = \frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(\tau^g)},$$

Zero profits in the innovation sector implies $q_{j+2}(\tau^g) = a * q_{j+1}(\tau^g) = a\frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(\tau^g)}$ which, as we have seen, is too high to make it profitable using capital $j+2$ in the consumption sector. As in the case of exogenous labor, the price of capital, $q_{j+1}(t)$, decreases during the innovation phase driving down $q_{j+2}(t)$ as well. However, with endogenous labor supply, $c(t)$ now increases

[41] Set $\zeta > \frac{a-1}{a-1/\gamma}$.

during the innovation phase that ends at $t = \tau^g + \tau^n$ when

$$q_{j+1}(\tau^g + \tau^n) = \frac{1}{a}\frac{\gamma - 1}{\gamma - 1/a}\frac{1}{bc(\tau^g + \tau^n)}.$$

The length of the innovation phase therefore satisfies

$$e^{(b-\rho)\tau^n} = a\frac{c(\tau^g + \tau^n)}{c(\tau^g)} = a\frac{l(\tau^g + \tau^n)}{l(\tau^g)}.$$

According to the results established for the baseline model, the wage rate, in units of the current consumption good, grows from $w(\tau^g) = \gamma^j \frac{a-1}{a-1/\gamma}$ at the beginning of an innovation phase, to $w(\tau^g + \tau^n) = \gamma^{j+1} \frac{a-1}{a-1/\gamma}$ at its end. On the other hand, during the innovation phase a combination of the production function and the first order condition w.r.t. labor supply implies that $w(t) = \zeta\gamma^j l(t)^{\frac{\eta}{\eta-1}}$. Hence

$$\frac{l(\tau^b + \tau^n)}{l(\tau^g)} = \gamma^{\frac{\eta-1}{\eta}}.$$

Notice that $l(\tau^g + \tau^n) = l(0)$, hence labor supply also follows a cycle. For later reference, denote $l^H \equiv l(0)$ and $l^L \equiv l(\tau^g)$, with $l^H > l^L$. The length of the innovation phase now is

$$\tau^n = \frac{\log a + \frac{\eta-1}{\eta}\log\gamma}{b-\rho},$$

which is longer than the value we computed under exogenous labor supply. Recall that $\tau^g = \frac{\log\gamma}{\eta(b-\rho)}$, with endogenous labor supply. The length of the whole cycle, however, remains unchanged,

$$\tau^g + \tau^n = \frac{\log\gamma}{\eta(b-\rho)} + \frac{\log a + \frac{\eta-1}{\eta}\log\gamma}{b-\rho} = \frac{\log a + \log\gamma}{b-\rho},$$

The growth rate of consumption and employment, during the innovation phase, satisfies

$$g^n = \frac{\frac{\eta-1}{\eta}\log\gamma}{\log a + \frac{\eta-1}{\eta}\log\gamma}(b-\rho).$$

This is smaller than the growth rate of consumption in the adoption phase, which is $b-\rho$. On the other hand, the LS behaves as in the exogenous labor supply case. That is, it decreases in the adoption phase and it increases in the innovation phase.

The capital stock is calculated in the same way as in the baseline model, the only difference being that we should now take into account the variation of total employment over time. Following the same procedure as before, one can show that the amount of capital $j+2$ at $t=\tau^g+\tau^m$ and the amount of capital $j+1$ at $t=0$ satisfy the following relation,

$$\frac{k_{j+2}(\tau^g+\tau^n)}{\gamma^{j+1}} = (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - \underline{x},$$

with $\underline{x}>0$ defined as

$$\underline{x} \equiv (a\gamma)^{\frac{\rho}{b-\rho}}[\frac{\tilde{\theta}_1}{\gamma^j}(1-\gamma^{-\frac{\rho/\eta}{b-\rho}}) + \frac{\tilde{\theta}_2}{\gamma^j}(1-\gamma^{-\frac{\eta b-(\eta-1)\rho}{\eta(b-\rho)}})] + \frac{\tilde{\theta}_3}{\gamma^j}a\gamma^{-\frac{1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}] + \gamma l^H,$$

and

$$\tilde{\theta}_1 \equiv \frac{\gamma^j l^H}{a\rho(\gamma-1)}[b(a\gamma-1)+\rho];$$
$$\tilde{\theta}_2 \equiv \frac{\gamma^j l^H}{a(\gamma-1)}\gamma\frac{ab+(b-\rho)(\eta-1)}{b-(\eta-1)(b-\rho)};$$
$$\tilde{\theta}_3 \equiv \frac{b}{b-g^b}\gamma^{j+1}l^L.$$

Inspection shows that $\underline{x}$ is independent of j, hence the relation between $\frac{k_{j+2}(t_{j+2})}{\gamma^{j+1}}$ and $\frac{k_{j+1}(t_{j+1})}{\gamma^j}$ is qualitatively the same as in the exogenous labor supply case, depicted in Figure 3.4. As $(a\gamma)^{\frac{\rho}{b-\rho}}>1$, there exists a unique steady state value of capital, k^*, as in the baseline model.

To evaluate the initial condition consider the case $0<k_0(0)<k_0^*\equiv ak^*+l^H$.[42] Denote with $k_0^1(0)$ the units of capital 0 used to produce the consumption good. Recall the production function is $c(t)=\min\{k_0^1(t), l_0(t)\}$. At $t=0$ the following equilibrium conditions hold

$$c=k_0^1=l_0;\quad 1=r+w;\quad w=\zeta l_0^{\frac{\eta}{\eta-1}}.$$

Given $k_0^1(0)$, the wage and rental rates are $w(0)=\zeta k_0^1(0)^{\frac{\eta}{\eta-1}}$, and $r(0)=1-w(0)$. The implied price of capital $j=0$ is

$$q_0(0)=[1-\zeta k_0^1(0)^{\frac{\eta}{\eta-1}}]\frac{1}{bc(0)}.$$

Denote with τ_1 the first time that capital $j=1$ is used to produce consumption. The price of

[42]Here k_0^* is the amount of capital $j=0$ when capital $j=1$ begins to be employed in production. Same as in the baseline model, how the economy behaves at $t=0$ for $k_0(0)>k_0^*$ can be determined from the path of capital stock derived for the case $0<k_0(0)<k_0^*$.

capital $j = 0$ for $t \in [0, \tau_1]$ satisfies

$$q_0(t) = [1 - \zeta k_0^1(t)^{\frac{\eta}{\eta-1}}] \frac{1}{bc(t)}.$$

On the other hand, we know from Proposition 1 that, at $t = \tau_1$,

$$q_0(\tau_1) = \frac{1}{a} \frac{\gamma - 1}{\gamma - 1/a} \frac{1}{bc(\tau_1)}.$$

It follows that $k_0^1(\tau_1) = [\frac{a-1}{a-1/\gamma}\frac{1}{\zeta}]^{\frac{\eta-1}{\eta}} = l^H$.

During the initial phase, $c(t) = k_0^1(t)$. Capital $j = 0$ self-accumulates during this phase, which implies that its price decreases over time at the rate $b - \rho$. That is, $q_0(t) = q_0(0)e^{-(b-\rho)t}$. Equivalently,

$$[k_0^1(t)^{-1} - \zeta k_0^1(t)^{\frac{1}{\eta-1}}] = [k_0^1(0)^{-1} - \zeta k_0^1(0)^{\frac{1}{\eta-1}}]e^{-(b-\rho)t}. \tag{1}$$

As $h(y) \equiv y^{-1} - \zeta y^{\frac{1}{\eta-1}}$ is a monotonically decreasing function in y, Equation (1) uniquely determines the value of $k_0^1(t)$ for a given $k_0^1(0)$.

The length of the initial phase, τ_1, is endogenously determined by the choice of the initial allocation. Specifically, from $q_0(\tau_1) = q_0(0)e^{-(b-\rho)\tau_1}$, τ_1 should satisfy

$$\tau_1 = \frac{1}{b-\rho} \log \left\{ a \frac{k_0^1(0)^{-1} - \zeta k_0^1(0)^{\frac{1}{\eta-1}}}{\frac{\gamma-1}{\gamma-1/a}\frac{1}{l^H}} \right\}.$$

The dynamics of $k_0(t)$ in the initial phase is

$$\dot{k}_0(t) = b[k_0(t) - k_0^1(t)], \quad \textit{for } t \in [0, \tau_1]$$

with $k_0^1(t)$ satisfying Equation (1). This ordinary differential equation, though not admitting an analytical solution, uniquely determines the value of $k_0(t)$ for $0 < t \le \tau_1$ given an initial value $k_0^1(0)$. At $t = \tau_1$, the following boundary condition holds,

$$k_1(\tau_1) = \frac{1}{a}[k_0(\tau_1) - l^H] = k^*.$$

where k^* is the steady state value of the normalized capital stock calculated above.

Note that a larger value of $k_0^1(0)$ leads to, (a), a larger $k_0^1(t), \forall t \in [0, \tau_1]$ and, consequently, a smaller $k_0(t) \forall t \in [0, \tau_1]$, and, (b), a smaller value of τ_1. As a result, with the choice of a larger initial $k_0^1(0)$ the value of $k_0(\tau_1)$ and $k_1(\tau_1)$ would be smaller. Therefore the l.h.s. of the last equation is a monotonically decreasing function of $k_0^1(0)$. Furthermore, when $k_0^1(0) \to 0$, the value of $\tau_1 \to \infty$ and $k_1(\tau_1) \to \infty > k^*$. When $k_0^1(0) \to k_0(0)$, $k_0(\tau_1) \to k_0(0)$ and the implied value of $k_1(\tau_1) < k^*$. These properties guarantee the existence of a unique value $k_0^1(0)$ satisfying the boundary condition above.

C.4 Real Output Growth

The Baseline Model In our model economy total output is composed of consumption and investment. Denote $p(t)$ the relative price of investment good to consumption good in time t, which is equal to the relative price of capital of type $j+1$ to the consumption good, $p_{j+1}(t)$, during an adoption phase in which capital j and $j+1$ are employed. During an adoption phase, $p(t)$ is a constant at $\frac{\gamma-1}{\gamma-1/a}\frac{1}{b}$. During an innovation phase, this relative price declines at the rate $b-\rho$. The growth rate of real output $(c(t)+p(t)i(t))$ is

$$\frac{c(t)}{c(t)+p(t)i(t)}\frac{\dot{c}(t)}{c(t)}+\frac{p(t)i(t)}{c(t)+p(t)i(t)}\frac{\dot{i}(t)}{i(t)}$$

This formula for the growth rate of total output, expressed in consumption units, use as base prices those of the first period. Without loss of generality, focus again on the growth cycle starting with the adoption phase using capital j and $j+1$ and normalize its initial time at $t=0$. During the adoption phase,

$$i(t)=\dot{k}(t)=b[k_{j+1}(t)+\frac{\gamma^{j+1}}{\gamma-1}]-\underbrace{\gamma^j\frac{b(a\gamma-1)+\rho}{a(\gamma-1)}e^{(b-\rho)t}}_{\iota(t)}$$

It follows that

$$\frac{\dot{i}(t)}{i(t)}=b-(b-\rho)\frac{\iota(t)}{\dot{k}(t)},$$

i.e. $i(t)$ grows at a rate smaller than b during the adoption phase, and this rate varies over time. Using the formula for capital stock derived earlier, one can obtain the expression of investment for $t\in[0,\tau^g]$. In particular, at $t=0,\tau^g$, we have

$$i(0)=\underbrace{bk_{j+1}(0)}_{i_1(0)}+\underbrace{\gamma^j\frac{b-\rho}{a(\gamma-1)}}_{i_2(0)},$$

and

$$i(\tau^g) = \underbrace{\frac{\gamma}{a^{\frac{\rho}{b-\rho}}} b k_{j+1}(0)}_{i_1(\tau^g)} + \underbrace{\gamma^{j+1} \frac{b-\rho}{a(\gamma-1)}}_{i_2(\tau^g)}$$

The growth rate from $i_1(0)$ to $i_1(\tau^g)$ is smaller than $b-\rho$, and the growth rate from $i_2(0)$ to $i_2(\tau^g)$ is $b-\rho$. We assume $i(\tau^g) > i(0)$, that is, the growth rate of investment is positive during the adoption phase.[43] Under this assumption, the growth rate of aggregate output during the adoption phase is greater than $\frac{c(t)}{c(t)+p^i(t)i(t)}(b-\rho)$.

During the innovation phase, the growth rate of consumption is zero. Investment satisfies

$$i(t) = \dot{k}_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}]$$

Therefore, investment grows at the rate b, and real output grows at the rate

$$\frac{p(t)i(t)}{c(t)+p(t)i(t)} b$$

Consumption grows at the rate $b-\rho$ during an adoption phase and at rate 0 during an innovation phase, while the growth rate of investment is larger during the innovation phase. In the data the growth rate of real output is higher when consumption grows faster. In our model this is equivalent to total output growing faster in the adoption than in the innovation phase, which is satisfied when the following parametric restriction holds:

$$\frac{b}{\rho} \frac{(a\gamma)^{\frac{\rho}{b-\rho}} - a^{\frac{\rho}{b-\rho}}}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1} < 1.$$

This is true whenever a is sufficiently greater than 1, i.e. when innovation is costly.

The extended model In the baseline model, investment grows at a slower rate in the adoption phase than in the innovation phase. This is not the case in the extended model. To see this point,

[43]Using the expression for $i(\tau^g)$ and $i(0)$, that $i(\tau^g) > i(0)$ requires the following restriction on parameters: $\frac{b(a\gamma-1)}{\rho(\gamma-1)} \frac{\gamma^{\frac{\rho}{b-\rho}}-1}{(a\gamma)^{\frac{\rho}{b-\rho}}-1} (a^{\frac{\rho}{b-\rho}} - \gamma) < 1$.

note that during an adoption phase when capital j and $j+1$ coexist,

$$\begin{aligned}
dk_{j+1}(t) &= bk^2_{j+1}(t)dt - k^3_{j+1}(t) + \frac{k^3_j(t)}{a} \\
&= b[k_{j+1}(t) - \gamma^{j+1}l(t)(1-\sigma_j(t))]dt - 0 - \gamma^j d[l(t)\sigma_j(t)]dt \\
&= bk_{j+1}(t)dt - \frac{\gamma^j l^H}{a(\gamma-1)}\left\{-[b(a\gamma-1)+\rho]e^{(b-\rho)t} + \gamma[ab+(b-\rho)(\eta-1)]e^{-(\eta-1)(b-\rho)t})]\right\}dt
\end{aligned}$$

Recall that $i(t) = \dot{k}_{j+1}(t)$. The growth rate of investment, $\dot{i}(t)/i(t)$ is

$$b - \frac{\gamma^j l^H}{\gamma-1}\frac{[(b-\rho)(b(a\gamma-1)+\rho)e^{(b-\rho)t} + (\eta-1)(b-\rho)\gamma(ab+(b-\rho)(\eta-1))e^{-(\eta-1)(b-\rho)t}]}{\dot{k}_{j+1}(t)}$$

During the following innovation phase,

$$\begin{aligned}
dk_{j+1}(t) &= b[k_{j+1}(t) - \gamma^{j+1}l(t)]dt \\
&= b[k_{j+1}(t) - \gamma^{j+1}l^L e^{g^b t}]dt
\end{aligned}$$

which implies that the growth rate of investment is

$$b - b\frac{\gamma^{j+1}l^L g^b e^{g^b t}}{\dot{k}_{j+1}(t)}$$

Comparison of the two growth rate formulas reveal that it cannot be established that the growth rate of investment in one phase is uniformly greater or smaller than in the other.

For the same reason, we cannot conclude that the growth of output is uniformly greater in one phase than in the other. We briefly discuss conditions to deliver the result that the *average* growth rate of output in the adoption phase is larger than in the innovation phase. Consider the growth cycle in which j and $j+1$ are employed, denote τ^j the beginning time of the cycle. Table C.1 shows the values of consumption at the beginning of the adoption phase, at the end of it, and at the beginning of the next adoption phase.

Note the relative price of investment, in terms of consumption at the three points in time in Table C.1, is always $\frac{a-1}{a-1/\gamma}\frac{1}{b}$. The average growth rate of output from $t = \tau_j$ to $t = \tau_{j+1}$ is $\frac{\log\gamma}{\log\gamma+\log a}(b-\rho)$. To obtain the result that the average output growth rate in the adoption phase is larger than that in the innovation phase, we only need to establish that the average output growth rate from $t = \tau_j$ to $t = \tau_j + \tau^g$ is greater than $\frac{\log\gamma}{\log\gamma+\log a}(b-\rho)$. As shown before, the

Table C.1: Values of Consumption and Investment

Time	Consumption	Investment
$t = \tau_j$	$\gamma^j \ell^H$	$bk_{j+1}(\tau_j)$
$t = \tau_j + \tau^g$	$\gamma^{j+1}\ell^L$	$b[k_{j+1}(\tau_j + \tau^g) - \gamma^{j+1}\ell^L]$
$t = \tau_{j+1}$	$\gamma^{j+1}\ell^H$	$bk_{j+2}(\tau_{j+1})$

growth rate of output is equal to the weighted average of growth rates in consumption and in investment

$$\frac{c(t)}{c(t)+p(t)i(t)}\frac{\dot{c}(t)}{c(t)} + \frac{p(t)i(t)}{c(t)+p(t)i(t)}\frac{\dot{i}(t)}{i(t)}$$

Note that consumption grows at the rate $b-\rho$ during the adoption phase. We further assume that the growth rate of investment is positive during an adoption phase.[44] Under this assumption, the growth rate expressed above is greater than $\frac{\log\gamma}{\log\gamma+\log a}(b-\rho)$ if

$$\frac{\ell^H}{\ell^H + \frac{k_{j+1}(\tau_j)}{\gamma^j}\frac{\gamma-1}{\gamma-1/a}} > \frac{\log\gamma}{\log\gamma+\log a}$$

where $\frac{k_{j+1}(\tau_j)}{\gamma^j}$ is the steady state value of normalized capital stock in the extended model. This condition is satisfied if $(a\gamma)^{\frac{\rho}{b-\rho}} - 1$ is sufficiently large (hence $\frac{k_{j+1}(\tau_j)}{\gamma^j}$) and a/γ is sufficiently greater than 1.

[44]We do not give the exact parametric conditions for this assumption to hold here as it is rather long and cumbersome. Investment at the beginning of the adoption phase is $bk_j + 1(\tau_j)$. Using the formula for capital stock, we can derive that the investment at the end of the adoption phase equals to $bk_{j+1}(\tau_j)\gamma^{\frac{b}{\eta(b-\rho)}}$ plus a term which might be positive a negative. The average growth rate of investment is positive if $\gamma^{\frac{b}{\eta(b-\rho)}}$ is sufficiently different from one.